Mathematics- A highway to the frontiers of Science

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Mr. President, fellow scientists, ladies and gentlemen ,

At the outset let me express my gratitude to the members of the sectional committee on mathematics for electing me to preside over the section at the 75th Science Congress. The mathematics section of the science congress is generally regarded as a forum of younger mathematicians , but since this is the Platinum Jubilee session, they decided to have a sectional president who would be as near to platinum jubilee in his age as was possible and perhaps that is how I came to be elected . Be that as it may, I regard it as an honour and I have accepted it as the token of love and regards which fellow mathematicians still retain for me. I thank the mathematical community again for the honour they have done me .

1. Introduction :

The theme of science congress this year is frontiers of science and technology and that saves me from a rather anomalous situation. Let me be frank. Mathematicians, in general, do not regard me as one of them, because I work on problems in theoretical Physics. At the most I would be regarded as a mathematics teacher, perhaps, a good mathematics teacher . On the other hand physicists would not take me to be one of them because they would consider me more oriented towards mathematics than towards physics . And both of them are correct in their own way. The fact is that I have been using my mathematics to understand the physical world and so the theme of the congress just fits in place and I shall try to illustrate the type of work I am doing under the heading- Mathematics, the Highway to the Frontiers of Science and Technology .

2. Newtonian Frame :

Newton was the pioneer in the use of the mathematical highway to reach and then to expand the frontiers of science. With his three laws of motion, his law of gravitation and his method of fluxions [which we now call differential calculus], he initiated the exploration along this high-way. Incidentially it was just last year (1987) that the world of science celebrated the 300th annivarsary of the publication of Newton's book " Principia " which contained all the information for exploration of this highway.

It is well-known that following the mathematical highway opened up by Newton, it was possible to expand the frontiers of Science and Technology in an unprecedented manner . Not only could we explain the phenomena of tides and eclipses, but also that of motions of the planets and discover new planets. In technology the entire edifice of present day civil and mechanical engineering is essentially Newtonion. Newton systematically explored the mathematics- high way and opened up several new frontiers of knowledge.

In spite of such phenomenal success of Newtonion frame work in understanding nature , there were many critics of his way of using the mathematical highway for exploring the frontiers of science and technology. The critism was mainly based on the following three points :

(1) Newton assumed the existance of a true even-flowing time, the same for all observers. This assumption essentially means that whether you are describing a terrestrial motion like that of the falling apple or a celestial phenomena like the motion of a planet round the sun, you use the same time coordinate t . The strongest criticism of this Newtonion assumption came from his contemporary

mathematician Leib~~u~~nitz. His arguments were quite simple. Suppose Newton's assumption is correct . Then there exists a single unique time-axis along which times of occurances of all events are marked . Thus there is apoint P on this time-axis which represents the instant when God created this universe . Now Leib~~u~~nitz argued, " Why should God Almighty follow Newton and create the universes precisely at this time and not a moment earlier or a moment later ? It is too much to put such restrictions on the ultimate Creator and so Newton's assumption cannot be accepted " . We may not agree with this type of arguements to-day, but one can clearly see the seeds of relativity of time in Leib~~u~~nitz arguements .

And after a lapses of about 200 years a correct formulation of this criticism was provided by Einstein's special theory of relativity which opened up a new express-way to explore the frontiers of knowledge .

We now turn to the second point of criticism ~~(2)~~ In his statementt of the law of gravitation Newton described the magnitude of gravitational attraction, he gave its direction but had nothing to say about the mechanism by which the attraction takes effect. As per Newton's law we say that the earth attracts the moon, but we have no~~t~~ answer to the querry " how is this attraction transmitted"? There is no~~t~~ rope tie~~d~~ between the earth and the moon . How does the earth pull the moon ? Newton had no answer to this criticism . He described this mechanism of attraction as " action at a distance ". In plain Hindi we can roughly translate Newton's motion of action at a distance as नज़रों से नज़रें मिलीं और हुआ आकर्षण. It is clear that such an explanation would not satisfy any one.

More than a century later a new way of transmitting a force was suggested by Farady. Maxwell developed this suggestion mathematically and thus opened up a newer highway of exploring nature through his field equations of the electromagnetic theory. We shall have a glimpse of this highway a little later in this talk .

We now take up the 3rd point of criticism, ~~(3)~~ Newton uses the term "mass " to describe two distinct properties of matter . In his second law of motion , he uses the term " mass " to measure the response that a body makes to an external force. White in his law of gravitation he uses the term "mass " to measure the amount of attractive force which a body can produce . So once the term mass is used to measure the capacity of a body to respond to a force and next the same term mass is used to measure the capcity of a body to produce a force. The criticism was that Newton had no mathematical arguements to expect the two masses to be equal . The mass appearing in the second law of motion is called the inertial mass M_i . The mass occuring in the law of gravitation is the gravitational mass M_g . Newton almost inadvertently (so to say) put down $M_i = M_g$. That in nature $M_i = M_g$ was first suggested by Galileo's famous experiment of dropping balls of different masses from the leaning tower of Pisa and is experimentally established to a satisfactory order by Oersted's experiments . But in constructing a mathematical theory the axioms have to be stated clearly and each axiom must have a role in the development of the mathematical theory. Newtonion theory of gravitation did not satisfy this basic axiomatic requirement.

But as the saying goes, proof of the pudding is in the eating. The Newtonion theory produced wonderful results . The mathematical frame-work worked and frontiers of science and technolgoy rapidly expanded . And with the expanding frontiers the limitations of Newtonion mathematical highway soon became apparent and the above criticisms paved the way to newer mathematical paths which would keep the frontiers moving. It is the saga of the changing mathematical highways which keep the frontiers of science and technology constantly expanding, that we now proceed to describe .

4. Maxwellian Field Theory :

When Coulomb described the action of one electric charge on another, he followed the Newtonion frame-work and described the magnitude of the force between them (as proportional to the product of the two charges) and the direction of this force. He described it as an "action at a distance phenomena". However the critics of this Newtonion concept had their way and Farady set about to describe how this action is transmitted . He assumed that a single charge, by its very presence, produces a disturbance in the space surrounding the charge. This disturbance spreads from point to point of the surrounding space. It is as if the charge produces its own sphere of influence, which goes on expanding . As soon as this sphere of influence reaches the other charges, it experiences an attraction (or a repulsion as the case may be) . This was, in essence, Farady's hypothesis about how the electric force between two charges ~~to~~ is transmitted from one point charge to another . It is clear that according to this hypothesis the action of one charge on the other is not transmitted instantaneously.

Maxwell picked up this hypothesis to construct a mathematical way based on it. The mathematical technique of partial differential equations was readily available to describe this transfer of influence from one point to a neighbouring point. The Farady-concept of field propagation , when translated in mathematical language by Maxwell gave rise to the differential equation.

$$\nabla^2 F - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0$$

for the propagation of the force-field F . Now this was a wave-equation, an equation already known from the propagation of sound waves and the constant ~~a~~ c represented the wave-velocity . From this mathematical treatment of Farady's hypothesis, Maxwell made two predictions .

(1) Electromagnetic effects are propagated in space as waves with a finite velocity which is equal to the velocity of light.

(2) Light is propagated as waves and light waves are electromagnetic waves .

Now if light waves are eletromagnetic waves then since light reaches us from distant stars one can conclude that electromagnetic waves would not need any medium [like a connecting wire] for transmission from one point to another.

This mathematical analysis of Maxwell led Hertz to devise an experiment wherein he would generate electromagnetic waves at one point of his laboratory and receive it at a distant point thus paving the way of wireless transmission of signals. Hertz lab. experiments created new science which in its turn was converted into new wireless communication technology by Marcony. Thus when we switch on a radio or a T.V set, we are using the new technology created by Marconi which was made possible by a new advance in science made by Hertz which in its turn was due to the mathematical approach to the problem by Maxwell. This is a very convincing illustration of Mathematics working as a highway towards frontiers of science and technology .

The role that mathematics played in the development of Science and Technology (ST) of telecommunications reminds me of a series of Science-booklets I read in my student-days. They were entitled T C Mitts learns mathematics, T .C.Mitts learns science and so on. The name T C.Mitts being formed out of the initial letters of " The Common Man In The Street" . The authors were a husband-wife pair, the husband, an artist would draw figures on a page and the wife would write explanatory notes on the opposite page. To explain the role of mathematics in the development of science and technology, they had drawn a picture of a 4 storey-building [ground + 3 floors]. On the ground floor are workshops and factories where technologists like Marconi work, Radio sets and TV sets are produced and people at large realise the progress of S and T . But this technology is based

~~ob~~ on labs. experimentation which is carried out on the first floor. There we have scientists~~like~~ like Hertz planning and performing scientific experiments. But these experimental scientists in their turn design their experiments to prove or disprove certain mathematical theories developed on the second floor. There theoretical scientists like Maxwell, following an axiomatic approach, begin with a plausible hypothesis, use known mathematical techniques to deduce certain conclusions. These conclusions are tested in the laboratories on the lower floor and if found experimentally varifable, they are transferred to the ground floor for generation of appropriate technology . And thus the caravan of progress of science and technology moves on .

But there was still another floor (the top floor) in Mitts' drawing . On this floor sit pure mathematicians who are engaged in developing mathematics for its own sake . What role ~~to~~ do they play in the activities going on in the lower floors of the edifice of progress in science and technology ? We now turn to describe a development based on the third point of criticism of Newtonion frame which clearly brings out the role played by pure mathematics in the progress of this caravan of science and technology.

5. Einsteins' thought-experiment :

The equality of gravitational and inertial mass implies that bodies of different masses falling under the gravitational field of the earth experience the same acceleration g . The force of attraction of the earth (gravitational mass E) on a body of gravitational mass M_g is $\frac{GEM_g}{a^2}$, where G is the gravitational constant and a is the radius of the earth . As a result the body experiences an acceleration f given by Newton's 2nd law of motion as

$$M_1 f = \text{acting force} = \frac{GE}{a^2} M_g .$$

If $M_i = M_g$, $f = \frac{GE}{a^2}$ a constant

Thus Newton's tacit assumption of equality of M_i and M_g is reflected in the constant acceleration due to gravity of all falling bodies.

Einstein, while working out his general theory of relativity had ~~to~~ two points to start with . One point was to base his theory explicitly on this axiom $M_i = M_g$ or its equivalent axiom of constancy of the acceleration of all falling bodies irrespective of their masses . The second point was, of course, the principle of relativity according to which laws of nature must be expressible in the same form by any two observers in relative motion with respect to each other. To understand how he went about doing this we shall describe, What has come to be known in literature as Einstein's thought-experiment .

Imagine a high-rise building, say a building with 100 floors and a lift starts downwards from the top floor. As the lift descends, assume that the connecting cable of the lift snaps so that it begins falling freely under gravity like the celebrated apple of Newton . You see why Einstein called it a thought experiment. It is not to be actually performed,only to be thought about .

Imagine further that there is a theoretician T_1 in the lift. He has , of course, his formulae $s = ut + \frac{1}{2}ft^2$, $v = u + ft$ etc. and so knowing the initial conditions he can work out the time t by which the lift will strike the ground , which will hardly ~~by~~ be a few seconds.Like any honest mathematician he decides to use these last few seconds of his life in an important scientific work. He decides to test Newton's law of gravitation in his lift. He takes out a piece of chalk from his pocket (poor mathematical scientists, what else would they generally find in their pockets except a chalk or two ?). He holds that piece of chalk in front of his eyes and then lets it go What does he find ? He would find that the piece of chalk does not "fall " down but remains poised in the air in front of his eyes . You see, if during a small time interval the chalk moves down a distance y , the floor of the lift also moves down through the same distance y and so the height of the chalk above the floor of the lift is unaltered. The chalk appears to be at rest to T_1 . T_1 concludesthat Newton's law of gravitation has ceased to apply to his chalk .

Note that this conclusion is a direct consequence of the equality of gravitational and inertial mass leading to the same acceleration of the chalk and the lift-floor .

For comparison let us now take another theoretician T_2 observing this accident of " freely falling lift " . He finds that (1) T_1 is falling freely with acceleration g , and that (2) the piece of chalk is also falling downwards towards the earth with the acceleration g . Or ~~t~~he findsthat Newton's law of gravitation applies perfectly to the piece of chalk .

So we have found two observers T_1 and T_2 in motion relative to each other who describe Newton's law of gravitation differently . But a law of nature must be describable in an invariant form by any two observers in relative motion . This means that Newton's description of gravitation does not satisfy the principle of relativity and so a new law obeying the relativity principle is called for. The gene~~r~~s~~is~~ of such a new invariant law of gravitation is also suggested by this thought-experiment.

Imagine now that T_1 gives a little horizontal [paralled to the lift-floor] push to the air-poised chalk. What will ~~be~~ he observe? He will observe that the piece of chalk moves uniformly in a straight line and strikesthe opposite wall of the lift. You see, the chalk travels a horizontol distance x, (because of the push) in any small time interval, but the vertical distance y covered by the chalk in this time interval is the same as the vertical distance y through which the floor of the lift falls and so the chalk appears to move in a straight line parallel to the floor of the lift .

~~But~~ description of the same phenomena by T_2 on the ground is different. He finds that the chalk piece is not moving in a straight line but in a parabola. The description of the gravitational phenomena by the two observers differ radically or rather differ geometrically . How ~~is~~ to reconcile this geometrical difference ?

Returning to the 4 storeyed edifice of science and technology built for T.C.Mitts we find that Einstein, a theoretical scientist working on the second floor wants to reconcile the principles of equality of gravitational and inertial mass with the principles of relativity and finds that the problem is geometrical . And geometry is the descipline of the top-floor . So let us now visit the top-floor of the science and technology building and witness the activities going on there .

6. Different Geometries :

It all began some two thousand years ago, when the Greek Mathematician Euclid collected all the results in geometry known in his time and arranged them as a logical sequence. Begining with certain undefined concepts and unproved axioms, he gave certain definitions, derived known geometrical results as a deductive chain of theorems, and thus created the science of geometry now known as Euclidean geometry. Euclid called his axioms as self-evident truths and most of them could be so described,as for example, " not more than one point can be common to two straight lines" or " all right angles are equal" . But then, there was one axiom, known as the parallel postulate or the 5th postulate which stated something like the following "Given a straight line l and a point P outside it, one and only one straight line can be drawn passing through P and parallel to l ". And this could, in no way, be regarded as self-evident truth ". It looks like any other theorem that Euclid deduced from his axioms . As a matter of fact, Euclid himself experienced a good deal of hesitation in accepting it as a postulate, even after spending considerable time and energy in unsuccessful attempts to deduce it from other axioms and theorems .

The hesitation which Euclid felt in including the parallel postulate in the list of his axioms of geometry, haunted several generations of geometers following Euclid and for almost a thousand years several mathematicians tried their luck in deducing this postulate as a theorem, but without any success. Ultimately in the 16th Century A.D a Polish mathematician Labochevsky overcame this hesitation and in a sense, established that Euclid's parallel postulate is an axiom . He negated this axiom i.e he replaced this parallel postulate of Euclid by a postulate which stated some thing like the following " Given a straight like 1 and a point P outside it at least two straight lines can be drawn through P and parallel to 1 " . He showed that this postulate works as well as Euclid's parallel postulate and one can deduce various theorems of geometry based on it.. Of course, these theorems are different from those of Euclidean geometry . Nevertheless they are equally true because to a mathematician true or false are relative terms implying consistency or otherwise of the logic he has used in deriving his results Thus a new geometry called Labochevskian geometry was created .

A couple of centuries roll by during which a number of mathematicians led by Gauss (the "Prince among mathematicians") studied geometry of curved surfaces and in so doing laid the foundations of a third geometrical descipline-. It was left to Riemann who in a famous paper of 1857 built the axiomatic structure of this third new geometry which came to be known as the Riemannian Geometry. He dropped two of the axioms of Euclidean geometry (1) regarding two straight lines not meeting in more than one point and (2) the parallel postulate . In Riemannian geometry two straight lines can have more than one point in common and the parallel postulate is replaced by a postulate which reads something like the following : "Give a straight line l and a point p outside it, no straight line canbbe drawn passing through p and parallel to l ".

All these activities were going on at the top-floor of the Science and technology building (as envisaged by T.C.Mitts) when Einstein working on a lower floor on theoretical problems of gravitation and relativity was faced with a situation which raised geometrical issues . We shall now see how Einstein choose a particular mathematical highway to explore the frontiers in his chosen field .

7. Riemannian High Way :

In order to appreciate Einstein's choice of Riemannian Geometry to describe the gravitational field in an invariant manner incorporating the axiom of equality of the two masses(gravitational and inertial), we shall remind ourselves of two relevant properties of this geometry . A~~t~~s a simple illustration of Riemannian geometry we take the geometry on the surface of a sphere. A straight line on the sphere is by definition the curve of shortest distance between the two points. We call it a geodesic and we know that geodesics on a sphere are great circles . Now any two great circleswill interest in a pair of points and not in a single point as in Euclidean geometry. Again since any two great circles must always intersect in diametrically opposite points we do not have parallel great circles and so the geometry on the surface of a sphere must be Riemannian .

The earth being roughly spherical in shape one must use Riemannian Geometry for measurements on the surface of the earth. But we know that for day to day surveying we use Euclidean geometry and not Riemannian and we explain it by saying that for ~~useful~~ usual surveying on a sportsfield, for example , the portion of the earth under consideration is so small compared to the size of the earth that for all practical purposes the curvature of the earth is neglible and we can regard it as flat in that small region and therefore use Euclidean geometry. Well this is a general property of the Riemannian geometry. In a region R of space where Riemannian geometry holds good, at every point P one can always choose a small neighbourhood N in which the geometry becomes Euclidean.

Turning now to the geometrical problem faced by Einstein in understanding ~~to~~ the outcome of his thought experiment ~~of Einstein~~, Einstein drew the following two axioms from the experiment :

(1) The geometry of the space in which a gravitational field is present is Riemannian.

(2) Freely falling bodies in this gravitational field describe geodesics of this Riemannian geometry .

We now try to see how both these axioms hold good for the two observers T_1 and T_2 . For T_1 in the falling lift, since his region of observation is limited to his lift, in this neighbourhood Euclidean geometry will apply and therefore he will not experience the presence of earth's gravitational field. Again geodesics in Euclidean space are straight lines and so he observes the piece of chalk as moving in a straight line.

But for the observer T_2 on the ground his range of observation is extensive and so the Riemannian nature of the space in the gravitational field asserts itself and he finds the chalk piece not moving in a straight line but in a curved path which must be the geodesics of the Riemannian space.

A little deeper thinking will suggest a ammendment to the above stated axiom (1). We mentioned above that for T_1 in the falling lift, the region of observation is limited to his immediate neighbourhood which is defined not only by the dimensions $\delta_x, \delta_y, \delta_z$ of the lift but also by the ~~line~~ time interval ~~straight~~ δt during which the lift is falling . You see, the neighbourhood is not the "lift" but the "falling lift". So Einstein's first axiom will now stand amended as follows :

(1) The space-time measurements in a region of space in which gravitational field is present follows a 4-dimensional Riemannian geometry.

Einstein later added a third axiom to determine the Riemannian geometry corresponding to the given sources of the gravitational field. There is no simple way of explaining how this third axiom was chosen, not simple enough to be included in an address of this type. And I do not think it is necessary for our purpose. Suffice it to say that Einstein was not geometer and it was his physical intuition which suggested to him that whether he liked it or not, exploring the frontiers of science of gravitation beyond Newtonion perceptions needed exploration along the mathematical highway opendd up by Riemann. He learnt geometry the hard way and followed the Riemannion high way. The results he achieved are so well known to all of us .

Let me end up by thanking all fellow mathematicians who have assembled here, especially those working on the top floor of the science and technology building for the help they render to persons like me who are shuttering between the top floor and the lower second floor.

Thank you .