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**GROUPS OF POINTS ON THE  
EQUI-ANHARMONIC CUBIC.**

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**1. Introduction.**

Every non-singular cubic can be reduced to the canonical form :

$$x^3 + y^3 + z^3 + 6mxyz = 0 \dots \dots \dots (i)$$

and if the cubic be equi-anharmonic we have the relation  $m=0$ , so that the standard form of the equi-anharmonic cubic can be taken as

$$x^3 + y^3 + z^3 = 0 \dots \dots \dots (ii)$$

In a former paper\* I have obtained (in collaboration with my friend Mr. M. Lakshmanamurthi) the complete set of necessary and sufficient conditions in order that any three points on the cubic (i) should be collinear. In attempting, as a natural consequence, an extension of this problem for the case of six points on the cubic (i) and finding out the conditions in order that they might form a group of zero residue, I found that the problem would be greatly simplified if I first tackled the case of the simpler cubic, *viz.*, the equi-anharmonic one whose equation is given by (ii).

I have given here all the results that I have obtained in this connection.

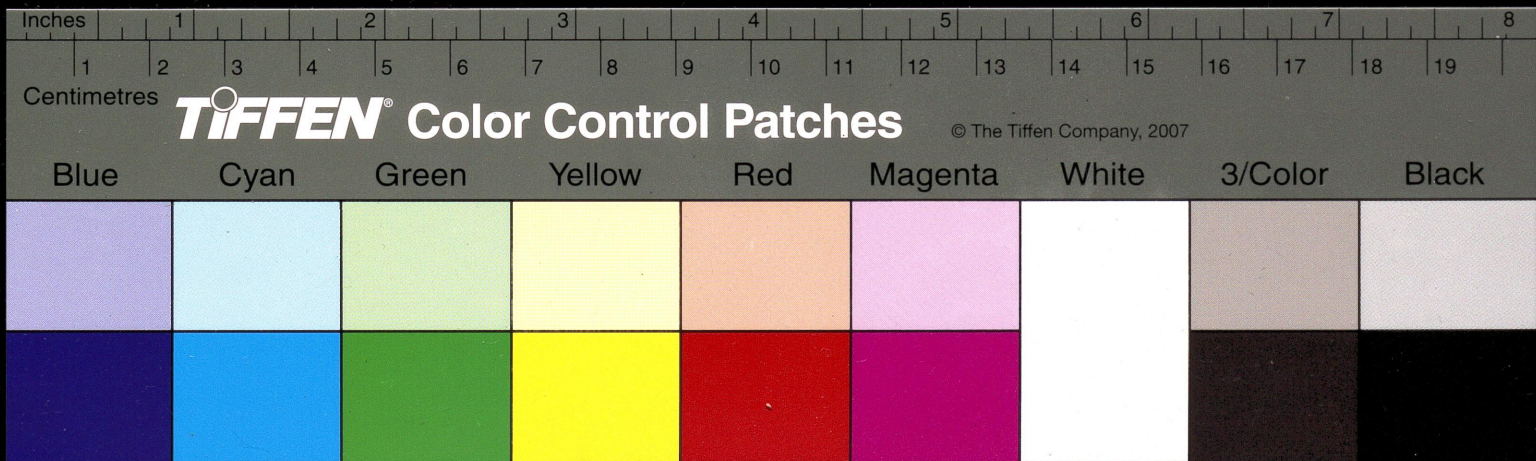
Further it is evident that since (ii) is only a particular case of (i) the problem of determining the condition of collinearity of three points on (ii) must yield simpler and a larger number of such conditions than in the case of (i). I have also indicated here these new sets of conditions obtained.

**2. Groups of Three Points.**

Let  $(x_1, y_1, z_1)_{1=1, 2, 3}$  be any three points on the cubic

$$x^3 + y^3 + z^3 = 0 \dots \dots \dots (ii)$$

\* Journal of the *Indian Mathematical Society*, February and April 1924.



and if they be collinear let them lie on the straight line

$$\alpha x + \beta y + \gamma z = 0 \quad \dots \quad \dots \quad \dots \quad (iii)$$

Associating with (ii) and (iii) another arbitrary relation

$$\rho x + q y + r z = 0 \quad \dots \quad \dots \quad \dots \quad (iv)$$

let us eliminate  $x : y : z$  between (ii), (iii) and (iv). The result of elimination reduces to

$$(\beta r - \gamma q) + (\gamma \rho - \alpha r)^3 + (\alpha q - \beta \rho)^3 = 0$$

which is a ternary cubic in  $(\rho, q, r)$  say

$$(A, B, C, F, G, H, I, J, K, L) (\rho, q, r)^3 = 0 \quad (v)$$

$$\begin{aligned} \text{where } -A &= \beta^3 - \gamma^3 & F &= \beta\gamma^2 & -I &= \beta^2\gamma \\ -B &= \gamma^3 - \alpha^3 & G &= \gamma\alpha^2 & -J &= \gamma^2\alpha \\ -C &= \alpha^3 - \beta^3 & H &= \alpha\beta^2 & -K &= \alpha^2\beta \end{aligned} \quad L = 0$$

But this eliminant can also be written in the form

$$(\rho x_1 + q y_1 + r z_1) (\rho x_2 + q y_2 + r z_2) (\rho x_3 + q y_3 + r z_3) = 0 \quad (vi)$$

so that we have the relations which express  $A, B$  and  $C$  in terms of the fundamental symmetrical functions of the system of roots  $(x_1, y_1, z_1)$ , etc. of (ii) and (iii), viz.,

$$\begin{aligned} A &= x_1 x_2 x_3 & 3F &= S y_1 y_2 z_3 & 3I &= S z_1 z_2 y_3 \\ B &= y_1 y_2 y_3 & 3G &= S z_1 z_2 x_3 & 3G &= S x_1 x_2 z_3 \\ C &= z_1 z_2 z_3 & 3H &= S x_1 x_2 y_3 & 3H &= S y_1 y_2 x_3 \end{aligned} \quad L = S x_1 y_2 z_3$$

where the symbol  $S$  denotes the sum of all the terms obtained by permuting the numbers 1, 2 and 3. Thus  $A, B, C, \dots, L$  are expressed in terms  $(x_1, y_1, z_1)$  as well as of  $(\alpha, \beta, \gamma)$ , viz.,

$$\begin{aligned} A &= x_1 x_2 x_3 = \gamma^3 - \beta^3 & 3F &= S y_1 y_2 z_3 = \beta\gamma^2 \\ B &= y_1 y_2 y_3 = \alpha^3 - \gamma^3 & 3G &= S z_1 z_2 x_3 = \gamma\alpha^2 \\ C &= z_1 z_2 z_3 = \beta^3 - \alpha^3 & 3H &= S x_1 x_2 y_3 = \alpha\beta^2 \\ 3I &= S z_1 z_2 y_3 = -\beta^2\gamma \\ 3J &= S x_1 x_2 z_3 = -\gamma^2\alpha \\ 3K &= S y_1 y_2 x_3 = -\alpha^2\beta \end{aligned} \quad L = S x_1 y_2 z_3 = 0$$

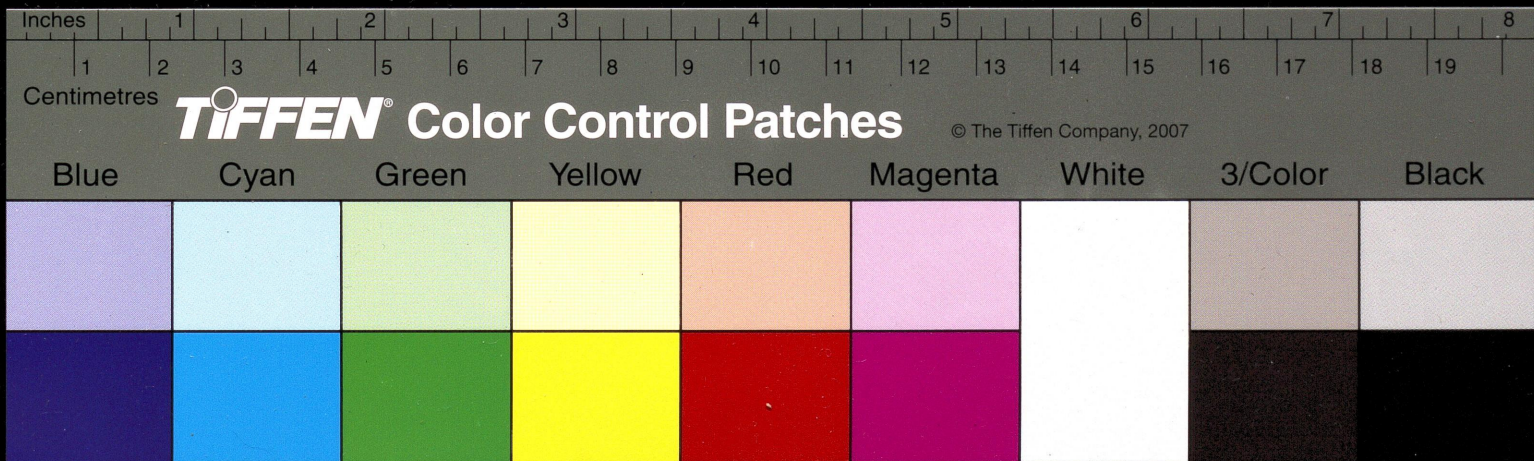
and eliminating  $\alpha : \beta : \gamma$  between any three of these expressions we shall have a necessary condition of collinearity.

We observe immediately that

$$6L = 0 = S x_1 y_2 z_3$$

so that  $\Delta x_3 (y_1 z_2 + y_2 z_1) = 0 \quad \dots \quad \dots \quad \dots \quad (vii)$

which is a necessary condition which holds in the general case (i) also.



Further  $A+B+C = \sum x_1 x_2 x_3 = \sum \gamma^3 - \beta^3 = 0 \dots$  (viii)  
 is also a necessary condition. In the paper referred to above we have shown that (vij) and (viii) taken together are both necessary and sufficient for the collinearity of the three points on the cubic (i) and hence also for (ii). It has also been shown, further, that of the three conditions

$$\left. \begin{aligned} x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3 &= 0 \\ x_1 y_2 z_3 + y_1 z_2 x_3 + z_1 x_2 y_3 &= 0 \\ x_1 y_3 z_2 + y_1 z_3 x_2 + z_1 x_3 y_2 &= 0 \end{aligned} \right\} \dots \text{(ix)}$$

any two taken together are necessary and sufficient conditions.

It is easy to see that the conditions in (ix) are all of the first degree in any one set of co-ordinates of  $(x_1, y_1, z_1)$ . In the case of the cubic (ii) we are considering, we can also deduce sets of conditions of the second degree a thing which is not present in the case of the cubic (i). We have

$$\begin{aligned} \frac{F}{I} &= -\frac{\gamma}{J} = \frac{G}{K} \\ \text{and } \frac{G}{J} &= -\frac{a}{\gamma} = \frac{H}{I} \\ \text{and } \frac{H}{K} &= -\frac{\beta}{a} = \frac{F}{J} \end{aligned}$$

which can be written in the form

$$FK = GI = HJ \dots \dots \dots \text{(x)}$$

equivalent to two conditions. On simplifying these conditions by putting

$$\begin{aligned} l &= x_1 x_2, m = y_1 y_2, n = z_1 z_2 \\ l' &= y_1 z_2 + y_2 z_1; m' = z_1 x_2 + z_2 x_1; n' = x_1 y_2 + x_2 y_1 \end{aligned}$$

they reduce to the forms

$$\begin{aligned} (mz_3 + l'y_3)(n'y_3 + mx_3) \\ = (nx_3 + m'z_3)(l'z_3 + ny_3) = (ly_3 + n'x_3)(m'x_3 + lz_3) \end{aligned}$$

### 3. Groups of Six Points.

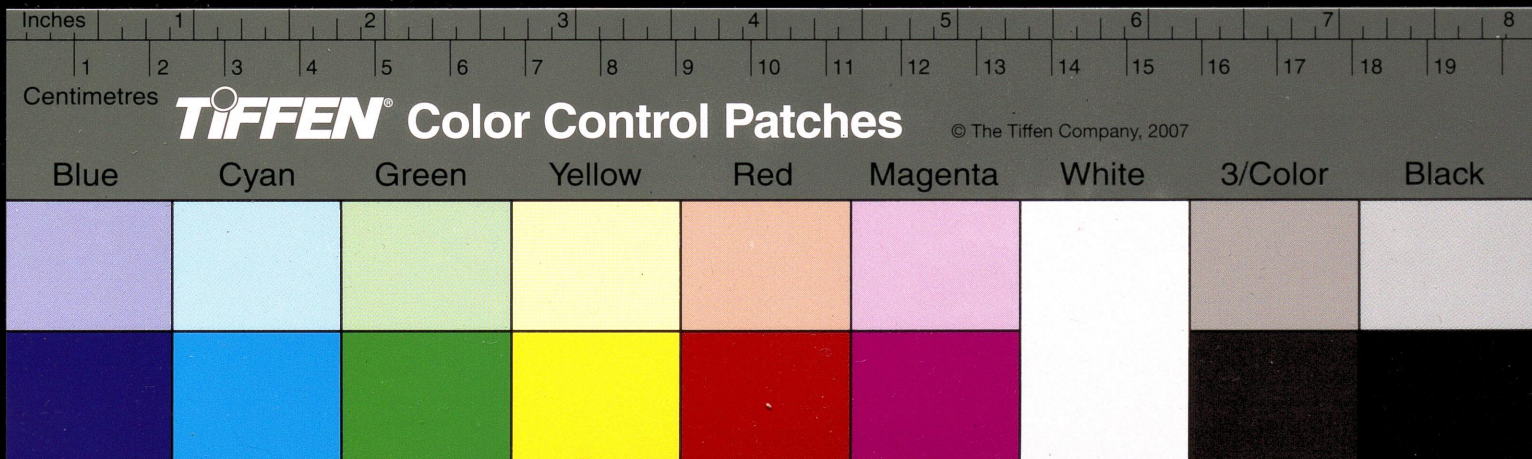
Let  $(x_1, y_1, z_1)_{1..6}$  be the intersections of the cubic  
 $x^3 + y^3 + z^3 = 0 \dots \dots \dots \text{(ii)}$

and a conic

$$(a_1 b_1 c_1 f_1 g_1 h) (x_1 y_1 z)^2 = 0 \dots \dots \text{(xi)}$$

Associating with these a linear condition

$$ax + \beta y + \gamma z = 0 \dots \dots \dots \text{(xii)}$$





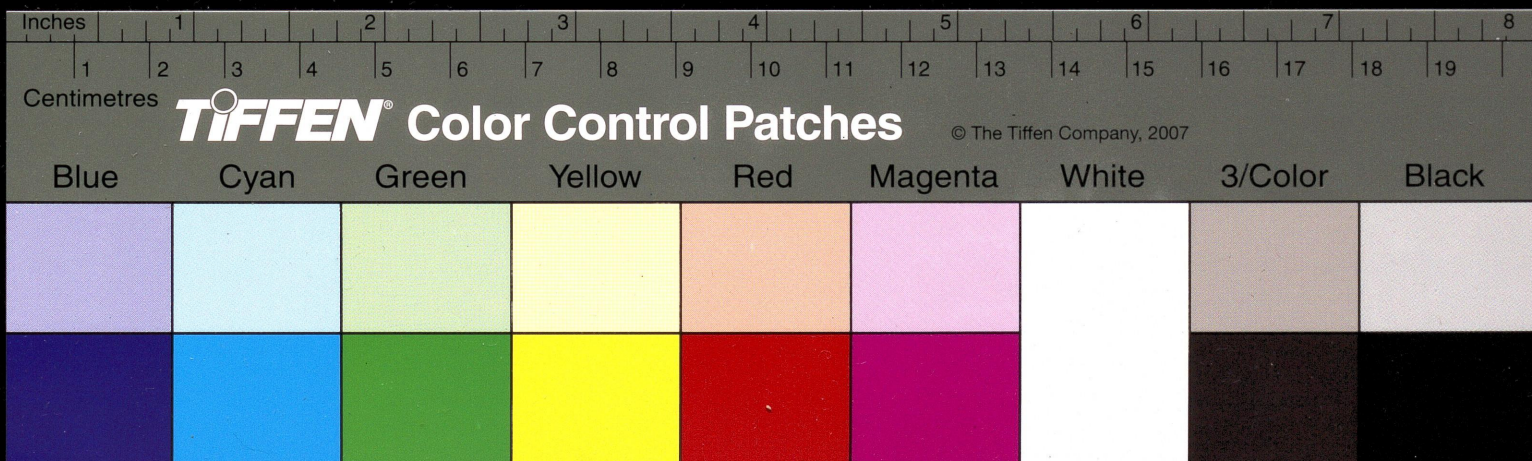
The fundamental symmetrical functions of the roots ( $X, Y, Z$ )  
 $f$  (ii) and (xii) are given by

$$\begin{array}{l|l} X_1 X_2 X_3 = A = \gamma^3 - \beta^3 & SY_1 Y_2 Z_3 = 3F = \beta\gamma^2 \\ Y_1 Y_2 Y_3 = B = \alpha^3 - \gamma^3 & SZ_1 Z_2 X_3 = 3G = \gamma\alpha^2 \\ Z_1 Z_2 Z_3 = C = \beta^3 - \alpha^3 & SX_1 X_2 Y_3 = 3H = \alpha\beta^2 \\ \\ SZ_1 Z_2 Y_3 = 3I = -\beta^2\gamma & \\ SX_1 X_2 Z_3 = 3J = -\gamma^2\alpha & SX_1 Y_2 Z_3 = L = 0 \\ SY_1 Y_2 X_3 = 3K = -\alpha^2\beta & \end{array}$$

and the symmetrical functions of the second degree which appear in  $E$  have, as already mentioned, been calculated by Cayley in terms of  $A, B, C, F, G, H, I, J, K$  and  $L$ . Substituting these values the eliminant  $E$  is further reduced to

$$\begin{aligned} E = & (\alpha^3 A^2 + \dots + \dots) \\ & + 8 (f^3 BC + \dots + \dots) \\ & + 24 (gh^2 AF + \dots + \dots) \\ & + 6 (b^2 f BF + \dots + \dots) \\ & + 12 (cf^2 CF + \dots + \dots) \\ & + 24 (g^2 h AI + \dots + \dots) \\ & + 12 (bf^2 BI + \dots + \dots) \\ & + 6 (c^2 f CI + \dots + \dots) \\ & + 24 (agh AL + \dots + \dots) \\ & + [b^2 c (9F^2 - 6BI) + \dots + \dots] \\ & + [bc^2 (9I^2 - 6CF) + \dots + \dots] \\ & + 4 [chf (9FG - 3CK) + \dots + \dots] \\ & + 4 [ahf (9JK - 3AF) + \dots + \dots] \\ & + 2 [bcf (9FI - 3BC) + \dots + \dots] \\ & + 24 fgh (FJ + GK + HI - L^2) \\ & + 12 [ch^2 (2FJ - GK - HI + L^2) + \dots + \dots] \\ & + 6 [bch (6FL - 3KI - BG) + \dots + \dots] \\ & + 6 [bcg (6IL - 3FG - CK) + \dots + \dots] \\ & + 6 abc (-FJ - GK - HI + 4L^2). \end{aligned}$$

Substituting the values of  $A, B, C, \dots, L$  in terms of  $\alpha, \beta$ , and  $\gamma$   
 $E$  finally reduces to a ternary sextic in  $\alpha, \beta$  and  $\gamma$  say  $E^v(\alpha, \beta, \gamma)$   
 whose co-efficients of functions of  $\alpha, \beta, \gamma$  are only functions of  $a, b, c, f, g, h$  only.









so that the resultant

$$\begin{aligned}
 E &= (1) + (2) + \dots + \dots + (19) \\
 \text{and } E_6^{-1} (a_1 \beta_1 \gamma) &= \pi (ax_1 + \beta g_1 + \gamma z_1) = \\
 & a^6 x_1 x_2 x_3 x_4 x_5 x_6 + \dots + \dots \\
 & + a^5 \beta^5 Sx_1 x_2 x_3 x_4 x_5 y_6 + \dots + \dots \\
 & + a^5 \beta^5 Sx_1 y_2 y_3 y_4 y_5 y_6 + \dots + \dots \\
 & + a^4 \beta^2 Sx_1 x_2 x_3 x_4 y_5 y_6 + \dots + \dots \\
 & + a^2 \beta^4 Sx_1 x_2 y_3 y_4 y_5 y_6 + \dots + \dots \\
 & + a^4 \beta \gamma Sx_1 x_2 x_3 x_4 y_5 z_6 + \dots + \dots \\
 & + a^2 \beta^3 Sx_1 x_2 x_3 y_4 y_5 y_6 + \dots + \dots \\
 & + a^0 \beta^2 \gamma Sx_1 x_2 x_3 y_4 y_5 z_6 + \dots + \dots \\
 & + a^3 \beta \gamma^2 Sx_1 x_2 x_3 y_4 z_5 z_6 + \dots + \dots \\
 & + a^2 \beta^2 \gamma^2 Sx_1 x_2 y_3 y_4 z_5 z_6 + \dots + \dots
 \end{aligned}$$

Hence comparing the co-efficients of the different terms in  $a^6, a^5 \beta, \dots, a^2 \beta^2 \gamma^2$  in  $E^{-1}$  and  $E = (1) + (2) + \dots + \dots - (19)$  we have the different symmetric functions in order, *viz.*,

- I.  $x_1 x_2 x_3 x_4 x_5 x_6 = b^3 + c^3 - 8f^3 + 9bcf$   
etc.
- II.  $Sx_1 x_2 x_3 x_4 x_5 y_6 = 6 (4f^2 g - 2chf - bcg - b^2 h)$   
etc.
- III.  $Sx_1 y_2 y_3 y_4 z_5 z_6 = 6 (4fg^2 - 2chg - acf - a^2 h)$   
etc.
- IV.  $Sx_1 x_2 x_3 x_4 y_5 y_6 = 3 (ab^2 + 2caf + 2cgh + 4bh^2 - 6fg^2)$   
etc.
- V.  $Sy_1 y_2 y_3 y_4 x_5 x_6 = 3 (ba^2 + 2bcg + 2chf + 4ah^2 - 6gf^2)$   
etc.
- VI.  $Sx_1 x_2 x_3 x_4 y_5 z_6 = 6 (abc - 4fgh - 4af^2 + 2bg^2 + 2ch^2)$   
etc.
- VII.  $Sx_1 x_2 x_3 y_4 y_5 y_6 = 8 (f^3 + g^3 + h^3) - 6 (bcf + cag + abh - 2c^3)$   
etc.
- VIII.  $Sx_1 x_2 x_3 y_4 y_5 z_6 = 6 (c^2 f + b^2 c + 6afg + 3cah)$   
etc.
- IX.  $Sx_1 x_2 x_3 y_4 z_5 z_6 = 6 (b^2 f + c^2 b + 6afh + 3bag)$   
etc.
- X.  $Sx_1 x_2 y_3 y_4 z_5 z_6 = 0.$

