

# COMMUTATION RULES RELATED TO PARTICLES OF SPINS HALF AND ONE

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## Introduction

For particles of spin half the relativistic wave equation can be put in the well-known form due to Dirac

$$\partial_\mu \beta_\mu \psi + \chi \psi = 0 \quad (1)$$

where the  $\beta$ -matrices satisfy the commutation rules

$$\frac{1}{2} (\beta_\mu \beta_\nu + \beta_\nu \beta_\mu) = \delta_{\mu\nu} \quad (2)$$

For particles of spins 0 and 1, Kemmer (1939) has given a theory including both types of particles by writing the wave equation in the form (1), but with the matrices satisfying the Duffin (1938) commutation rules

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\mu\nu} \quad (3)$$

In a previous paper (Madhava Rao, 1942, referred to here as A) we have investigated whether a similar theory can also be set up for particles of higher spins, and shown that commutation rules analogous to (2) and (3) can also be set up for such particles. These rules however [Equations (26) and (34) of A.] involve an arbitrary numerical constant  $k$  and it appears that its unique determination necessitates a further assumption besides those used in A for finding the commutation rules.

We apply in this paper the methods of A to derive the commutation rules (2) and (3) for particles of spins  $1/2$ , and 0 and 1 respectively, and show that the arbitrary numerical constant  $k$  is determined, in these cases of lower spins, by the assumption that one should be able to pass on to the second order wave equation from the first order wave equation (1).

## Method of obtaining the Commutation rules

Using the notation of A, the assumptions made in deriving the commutation rules can be stated as follows:—

(1) The wave equation is relativistic invariant. This leads to the condition

$$\beta_\mu t_{\nu\rho} - t_{\nu\rho} \beta_\mu = \delta_{\mu\nu} \beta_\rho - \delta_{\mu\rho} \beta_\nu \quad (4)$$

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where

$$t_{\mu\nu} = i S_{\mu\nu}, \text{ and } S_{\mu\nu} \text{ is the spin operator.}$$

(2)  $t_{\mu\nu}$  (or  $S_{\mu\nu}$ ) can be expressed in the form

$$t_{\mu\nu} = k (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \quad (5)$$

where  $k$  is an arbitrary numerical constant.

These two assumptions are valid for all spins, and (4) and (5) can be combined into the single equation

$$k (\beta_\mu, [\beta_\nu, \beta_\rho]) = \delta_{\mu\nu} \beta_\rho - \delta_{\mu\rho} \beta_\nu \quad (\nu \neq \rho) \quad (6)$$

the brackets on the left-hand side being Poisson brackets. Taking the cases  $\mu = \nu \neq \rho$ , and  $\mu \neq \nu \neq \rho$ , equation (6) can be written in the equivalent forms

$$\left. \begin{aligned} (\beta_\nu, t_{\nu\rho}) &= \beta_\rho \\ \text{or, } \beta_\rho &= k (\beta_\rho \beta_\nu^2 - 2 \beta_\nu \beta_\rho \beta_\nu + \beta_\nu^2 \beta_\rho) \end{aligned} \right\} \quad (6, a)$$

and

$$\left. \begin{aligned} (\beta_\mu, t_{\nu\rho}) &= (\beta_\nu, t_{\mu\rho}) = 0 \\ \text{or, } \beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu &= \beta_\nu \beta_\rho \beta_\mu + \beta_\mu \beta_\rho \beta_\nu = \beta_\rho \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\rho \\ (\mu \neq \nu \neq \rho) \end{aligned} \right\} \quad (6, b)$$

We also write

$$\gamma_\lambda = \beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu \quad (6, c)$$

where  $\lambda, \mu, \nu, \rho$  are all different.

(3) The third assumption is that the spin operator  $S_{\mu\nu}$  satisfies an algebraic equation whose roots are the eigen-values of any component of the operator. For the cases considered in this paper these eigen-values are

$$-1/2, 0, +1/2. \quad \text{for spin } \frac{1}{2}.$$

$$-1, 0, +1 \quad \text{for spin } 1.$$

and the corresponding equations for  $t_{\mu\nu}$  are

$$t_{\mu\nu}^2 + \frac{1}{4} = 0 \quad \text{for spin } \frac{1}{2} \quad (7, a)$$

$$t_{\mu\nu}^3 + t_{\mu\nu} = 0 \quad \text{for spin } 1 \quad (7, b)$$

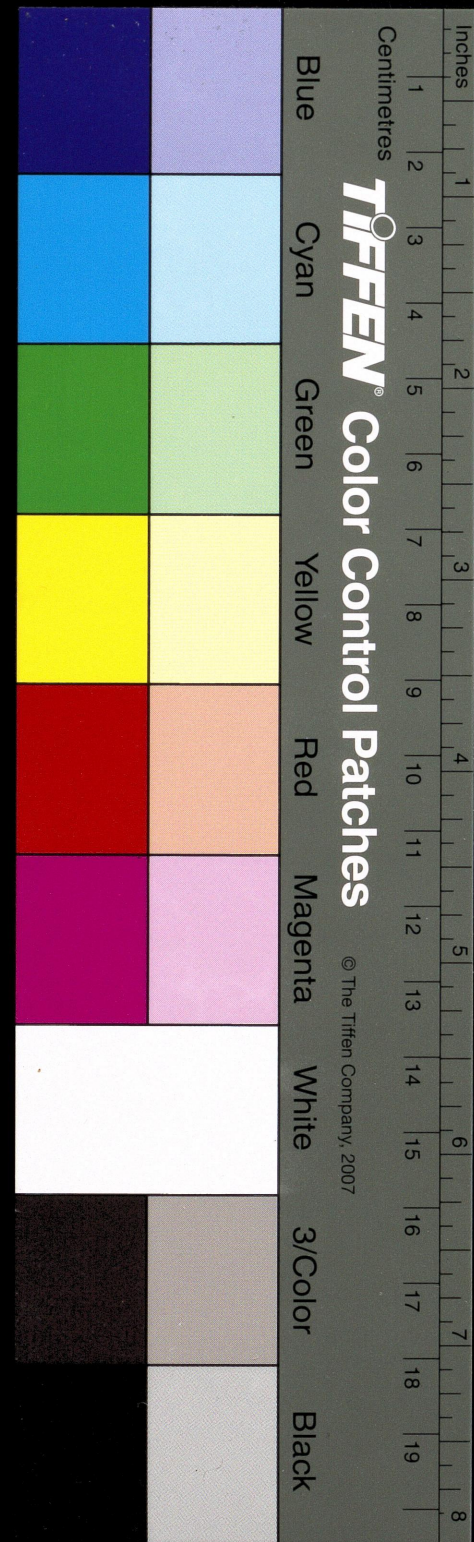
#### Case of spin 1/2

Taking the Poisson bracket of (7, a) with  $\beta_\nu$  twice successively, and using (5) and (6, a) we get

$$\beta_\mu^2 = \frac{1}{4} k \quad (8)$$

Taking the P.B. (abbreviation for Poisson bracket) of (8) with  $t_{\mu\nu}$ , and using (6, a) we get

$$\beta_\mu \beta_\nu + \beta_\nu \beta_\mu = 0 \quad (\mu \neq \nu) \quad (9)$$



(8) and (9) are the commutation rules for this case, and can be combined into the single rule

$$\beta_\mu \beta_\nu + \beta_\nu \beta_\mu = \frac{1}{2k} \delta_{\mu\nu} \quad (10)$$

We shall now show that the arbitrary constant  $k$  is uniquely determined by the condition that we can go to the second order wave equation from (1) by using the rule (10). Multiplying (1) throughout by  $\delta_\nu \beta_\nu$  on the left we get

$$\partial_\nu \partial_\mu \beta_\nu \beta_\mu \psi + \chi \partial_\nu \beta_\nu \psi = 0$$

or, using (10) and (1)

$$\partial_\nu \partial_\mu \left( \frac{1}{2k} \delta_{\mu\nu} - \beta_\mu \beta_\nu \right) \psi + \chi \partial_\nu \beta_\nu \psi = 0$$

i.e.,

$$\frac{1}{2k} \partial_\mu \partial_\mu \psi = 2 \chi^2 \psi$$

This will be of the form of the second order wave equation

$$\partial_\mu \partial_\mu \psi = \chi^2 \psi \quad (11)$$

provided  $k = 1/4$ , and the commutation rule (10) then reduces to the form (2); also equation (5) gives

$$t_{\mu\nu} = \frac{1}{4} (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \quad (5, a)$$

$$\text{or } S_{\mu\nu} = \frac{1}{4i} (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu)$$

#### Case of spin 1 (including 0)

The commutation rules in this case consist in the expression of groups of terms which are products of the  $\beta$ 's of the third degree in terms of those of lower order. We require therefore the use of three indices which we denote by  $\mu, \nu, \rho$ . Three cases arise according as the indices are repeated or not as follows:—

- (a) all the indices equal,
- (b) two indices equal, and one different,
- (c) all three indices equal.

We now take the P.B. of both sides of equation (7, b) with  $\beta_\nu$  three times successively, and simplify with the aid of (5) and (6, a) and (7, b). This gives, after some calculation,

$$k \beta_\mu^3 = \beta_\mu \quad (12)$$

which is the rule for the case (a).



Next taking the P.B. of (12) with  $t_{\mu\nu}$  ( $\mu \neq \nu$ ), we get using (6, a)

$$k(\beta_\mu^2 \beta_\nu + \beta_\mu \beta_\nu \beta_\mu + \beta_\nu \beta_\mu^2) = \beta_\nu$$

Combining this with (6, a), i.e.,

$$k(\beta_\mu^2 \beta_\nu - 2\beta_\mu \beta_\nu \beta_\mu + \beta_\nu \beta_\mu^2) = \beta_\nu$$

we derive the rules

$$k(\beta_\mu^2 \beta_\nu + \beta_\nu \beta_\mu^2) = \beta_\nu \quad (13)$$

and,

$$\beta_\mu \beta_\nu \beta_\mu = 0 \quad (14)$$

(13) and (14) are the commutation rules for the case (b). Finally take the P.B. of (14) with  $t_{\mu\rho}$  ( $\mu \neq \nu \neq \rho$ ), and using (6, a) and (6, b) we get easily

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = 0, \quad (\mu \neq \nu \neq \rho) \quad (15)$$

or with (6, c)  $\gamma_\lambda = 0$ ,

which is the commutation rule for the case (c). It can be easily shown that we get the same rule (15) if we start with (13) and take the P.B. with  $t_{\mu\rho}$  keeping, however, (6, c) in mind.

The rules (12), (13), (14), and (15) can be combined into the single rule

$$k(\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu) = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\mu\nu} \quad (16)$$

To reduce (1) to the second order we multiply it by  $k\partial_\rho \beta_\rho \beta_\nu$  on the left getting

$$k\partial_\rho \partial_\mu \beta_\rho \beta_\nu \beta_\mu \psi + k\chi \partial_\rho \beta_\rho \beta_\nu \psi = 0$$

or, using (16)

$$\partial_\rho \partial_\mu (\beta_\rho \delta_{\mu\nu} + \beta_\mu \delta_{\nu\rho} - k\beta_\mu \beta_\nu \beta_\rho) \psi + k\chi \partial_\rho \beta_\rho \beta_\nu \psi = 0$$

$$\text{i.e.,} \quad \partial_\rho \partial_\nu \beta_\rho \psi + \partial_\nu \partial_\mu \beta_\mu \psi = k\partial_\rho \partial_\mu \beta_\mu \beta_\nu \beta_\rho \psi + k\chi \partial_\rho \beta_\rho \beta_\nu \psi = 0$$

$$\text{or,} \quad -\chi \partial_\nu \psi - \chi \partial_\nu \psi + k\chi \partial_\mu \beta_\mu \beta_\nu \psi + k\chi \partial_\rho \beta_\rho \beta_\nu \psi = 0$$

$$\partial_\nu \psi = k\partial_\mu \beta_\mu \beta_\nu \psi$$

i.e.,

$$\partial_\mu \psi = k\partial_\nu \beta_\nu \beta_\mu \psi$$

Multiplying again by  $\partial_\mu$  on both sides

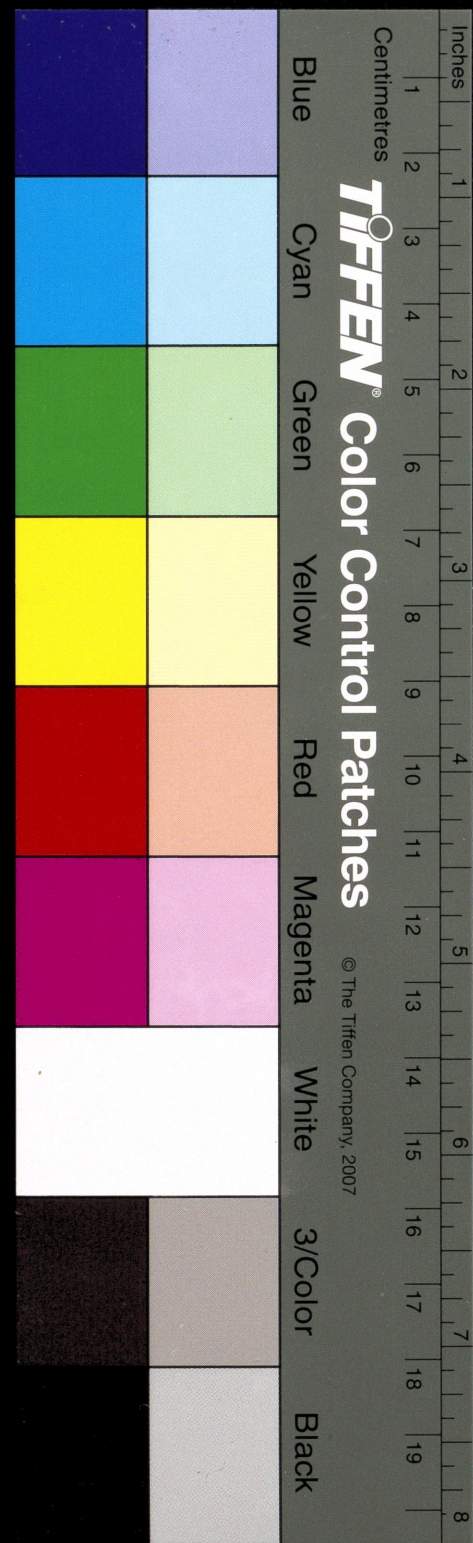
$$\partial_\mu \partial_\mu \psi = k\partial_\mu \partial_\nu \beta_\nu \beta_\mu \psi = kx^2 \psi$$

Reduction to the second order equation requires therefore  $k=1$ , and (16) then reduces to (3), and equation (5) gives

$$t_{\mu\nu} = (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu)$$

or,

$$S_{\mu\nu} = \frac{1}{i} (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \quad (5, b)$$

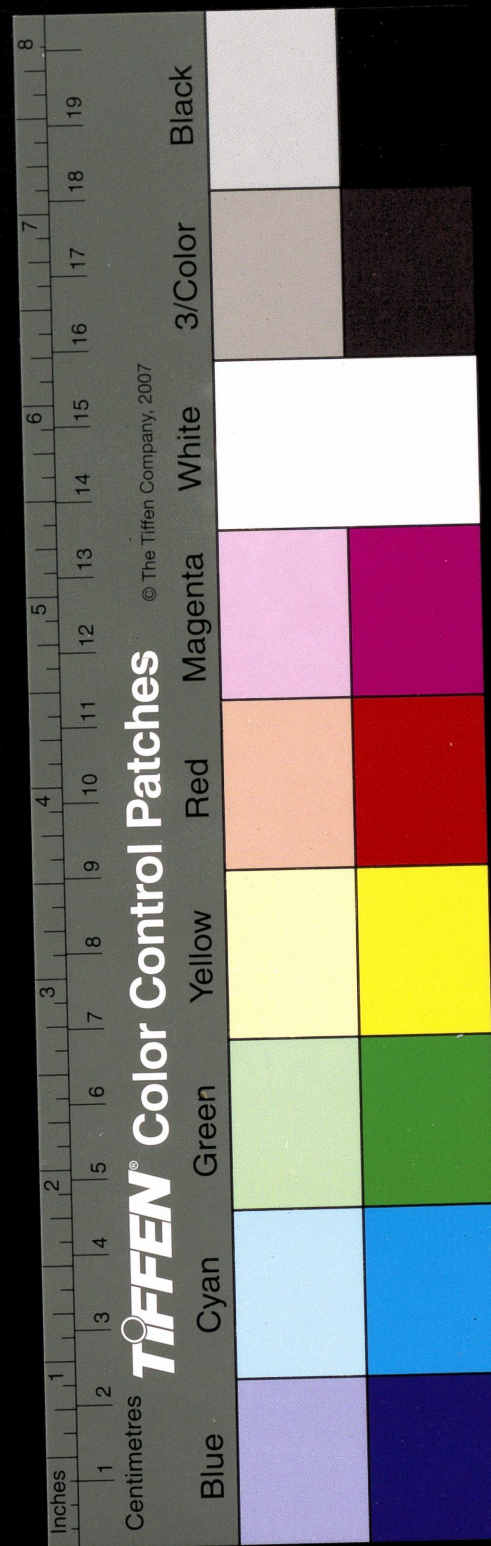


*Conclusion*

The application of the above method of the reduction of the first order wave equation to the second order one can be shown to lead, in the case of the higher spins  $3/2$  and  $2$ , to *two* alternative values of  $k$ , unlike in the case of the lower spins treated in this paper where  $k$  is uniquely determined. It appears therefore that some further or alternative criterion is necessary to determine  $k$  uniquely in the case of higher spin particles. This will be dealt with in a later paper.

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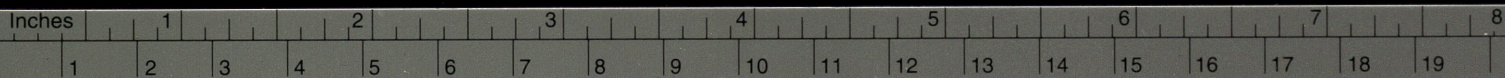
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