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Some inequalities relating to Hermite orthogonal functions.

by

B.S.Madhavarao

1. Abstract. As is well known, the Hermite orthogonal functions are the eigen-functions of the quantum-mechanical problem of the linear ~~har~~ harmonic oscillator. Using this, and the notions of probability density for the quantum and classical states, and the correspondence principle, a known inequality relating to the Hermite functions has been employed to derive two new types of inequalities, one referring to the largest zeros of this function, and the other to the largest zeros of its derivative. The first inequality has been proved, and the other is presented here as a conjecture. A table calculated for $n = 1(1)10$, 2D is given to show the plausibility of this conjecture.

2. Introduction. The normalised eigen-function of the ~~H~~ harmonic oscillator is given by

$$\phi_n(x) = \left(\pi^{1/2} 2^n n!\right)^{-1/2} e^{-x^2/2} H_n(x) \tag{1}$$

where $H_n(x)$ is the Hermite polynomial satisfying the recurrence relation

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x), \quad n \geq 1 \tag{2}$$

Using this, and $H_0(x) = 1$, $H_1(x) = 2x$, one can derive expressions for for higher values of n.

Denoting the successive relative maxima of $|\phi_n(x)|$ as x decreases from $+\infty$ to 0 by

$$\mu_{0,n},\ \mu_{1,n},\ \mu_{2,n},\ \cdots \tag{3}$$

it follows from the well-known theorem of Sonine[1] relating to orthogonal polynomials, that the sequence (3) is decreasing, or

$$\mu_{r,n} > \mu_{r+1,n} \quad (r = 0, 1, \cdots) \tag{3,a}$$

We shall use in this paper the further inequality due to Szasz[2] that

$$\mu_{r,n} > \mu_{r,n+1}, \quad (n \geq r \geq 0) \tag{4}$$

which relates to the corresponding maxima of ϕ_n and ϕ_{n+1}. Alternatively, we can put (4) in the form[3]

$$\operatorname{Max} |\phi_n(x)| \leq \operatorname{Max} |\phi_0(x)| = \pi^{-1/4} \tag{4,a}$$

Since we are interested ~~in the~~ with the highest relative maximum ~~in~~ we shall use (4) in the form

$$\mu_{0,n}^2 > \mu_{0,n+1}^2 \tag{5}$$

3. Zeros of ϕ_n and ϕ_{n+1}, and ϕ'_n & ϕ'_{n+1}. Before proceeding further we shall summarise, for the sake of easy reference, some of the well-known results about these zeros. The zeros of $\phi_n(x)$ are as is evident from (1) the zeros of $H_n(x)$, and it is known that $H_n(x)$ has n zeros all real and simple, say

$$x_{1,n} > x_{2,n} > \cdots x_{n,n} \tag{6}$$

and in addition $\phi_n(x)$ vanishes at $x = \pm\infty$. We also have the standard formulae for the derivative of $H_n(x)$ viz.

$$H_n'(x) = 2n\, H_{n-1}(x) \qquad (7)$$

$$H_n(x) = 2x\, H_{n-1}(x) - H_{n-1}'(x) \qquad (7,a)$$

so that

$$H_n'(x) = 2x\, H_n(x) - H_{n+1}(x) \qquad (8)$$

Differentiation of (1) shows that the zeros of ϕ_n' are given by the zeros of

$$H_n'(x) - x\, H_n(x) = 0 \qquad (9)$$

or, using (8) those of $x H_n(x) - H_{n+1}(x) = 0$ (9,a)

By Rolle's theorem, between any two roots of $\phi_n(x) = 0$ lies at least one root of $\phi_n'(x) = 0$, so that, as is evident from (9) or (9,a), $\phi_n'(x) = 0$ has $(n+1)$ roots, say

$$x'_{0,n} > x'_{1,n} > \cdots > x'_{n,n} \qquad (10)$$

and in addition $\phi_n'(x)$ vanishes at $x = \pm\infty$. Thus,

$$x'_{0,n} > x_{1,n} > x'_{1,n} > x_{2,n} > \cdots > x_{n,n} > x'_{n,n} \qquad (11)$$

and simðlarly,

$$x'_{0,n+1} > x_{1,n+1} > x'_{1,n+1} > x_{2,n+1} > \cdots \qquad (11,a)$$

Again (7) shows that the roots of $H_n(x)$ are the roots of $H'_{n+1}(x)$, and are interlaced with the roots of $H_{n+1}(x)$, thus

$$x_{1,n+1} > x_{1,n} > x_{2,n+1} > x_{2,n} > \cdots > x_{n,n+1} > x_{n,n} > x_{n+1,n+1} \qquad (12)$$

Hence, $x'_{0,n+1} > x_{1,n+1} > x_{1,n}$ & $x'_{0,n} > x_{1,n} > x'_{1,n}$ (13)

Also, $x'_{0,n+1} > x'_{0,n} > x'_{1,n+1} > x'_{1,n} > \cdots > x'_{n+1,n+1}$ (13,a)

These ~~The~~ inequalities give all the information re. the location of the roots of ϕ_n, ϕ_{n+1}, ϕ_n' & ϕ_{n+1}', and how they are interlaced.

As regards actual expressions for the roots of $\phi_n(x)$ i.e of $H_n(x)$, it might be noted that although this ~~a~~ is a polynomial equation in $x^2_{r,n}$ $(r = 1, 2, \ldots n)$ (taking out the root $x = 0$ for the case <u>n</u> odd), there are no direct expressions for $x^2_{r,n}$ itself to be found in the literature. However, there are some very interesting results relating to $x_{r,n}$. Thus, one has the inequality[4]

$$x_{r,n} < (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6}\, i_r \qquad (14)$$

where $i_1 < i_2 < i_3 < \cdots$ $(i_1 > 0)$ are the real zeros of the Airy's function $A(x)$. Further, for a fixed <u>r</u>,

$$x_{r,n} = (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6}\{i_r + \epsilon_n\} \qquad (15)$$

where $\lim_{n\to\infty} \epsilon_n = 0$. It follows from (14) and (15) that ~~the~~ the constant i_r is the best possible if <u>r</u> is fixed, and <u>n</u> arbitrary, and also that $(2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6}\, i_1$ is an upper bound for the zeros of $H_n(x)$ with the constant

$$6^{-1/3}\, i_1 = 1.85575\ldots \qquad (16)$$

not being irreplceable by a smaller one. An alternative form for (15) is given by

$$x_{r,n} = (2n)^{1/2} - 6^{-1/3}\, i_r\, (2n)^{-1/6} + o\left(n^{-1/6}\right) \qquad (15,a)$$

From the computational point of view, we might mention the tabulation by ~~H.E.~~ Salzer, Zucker & Capuano[5] of the largest zeros of $H_n(x)$ for $n = 1(13)15$D,

$\underline{n}$ = (14-16)14D, and $\underline{n}$ = (17-20)13D.

As regards the zeros of $\phi_n'(x)$, no information either regarding explicit expressions, or computational approach is available in the literature, except, perhaps, for the general theorem[6] that if ~~if~~ $\phi_n(x)$ and $\phi_n'(x)$ are both orthogonal systems, then $\phi_n(x)$ must be the system of Jacobi, Laguerre, or Hermite polynomials.

4. Quantum-theoretic considerations. We use the physical interpretation based on the notion of probability that $\phi_n^2\,dx$ defines the probability of occurence of the particle in the element of distance dx, or in other words, that ϕ_n^2 is the probability density or probability amplitude at $\underline{x}$. The corresponding probability distribution function ~~for~~ for the classical harmonic oscillator (of frequency ν_0, and total energy given by the eigen-value $E_n = h\nu_0(n+1/2)$ corresponding to the eigen-function ϕ_n) is given by

$$P\,dx = \frac{dx}{\pi\sqrt{(2n+1-x^2)}} \tag{17}$$

Equation (17) shows that for the classical oscillator, the probability density increases from $1/\pi\sqrt{(2n+1)}$ at $x=0$ to ∞ at $x = x_0 = \sqrt{(2n+1)}$

Also from the Schrodinger equation

$$\frac{d^2\phi}{dx^2} + (2n+1-x^2)\,\phi = 0 \tag{18}$$

we have $\frac{d^2\phi}{dx^2} = 0$ for $x = x_0$ showing that this constitutes a point of inflexion. For values of $|x| > |x_0|$, ϕ decrases continuously without exhibiting any nodes.

Plotting ϕ_n^2 against $\underline{x}$ (for different values of $\underline{n}$), we get the quantum probability distribution curves (denoted by A) which indicate the statistical interpretation of the Schrodinger equation in consonance with the principle of indeterminacy. Using equation (17) the classical distribution curves (denoted by B) can also be drawn along with the curves A for different values of $\underline{n}$. Such curves A and B are to be found in most of the standard text-books[7,8] on quantum mechanics, generally for values of n = 0(1)4. However, using the table of Hermite functions given by J.B.Russell[9], we can draw the curves for higher values of $\underline{n} > 4$. In view of the fact that we are going to refer to these for use only in two cases, it has been thought ~~been~~ not worthwhile to indicate these here. Since similar curves symmetrical with respect to the ϕ^2-axis may be plotted for negative values of $\underline{x}$ also (the curves plotted in the text-books being plotted for + ve $\underline{x}$), and since the ϕ's are normalised, we have the result that the total area under the curve for $-\infty \leq x \leq +\infty$ is equal to 1. These curves also ~~illust~~ illustrate the results regarding the location of the roots of $\phi_n, \phi_{n+1}, \phi'_n, \phi'_{n+1}$ as given by (10), (11), (12) & (13), and the inequality (5). In particular, we might interpret (5) by saying that the maximum probability density decreases as the quantum number n increases. This is in consonance with the correspondence principle, but this argument is, of course, no substitute for the rigorous mathematical deduction of (5).

5. Inequality relating to zeros of $\phi_n(x)$ ~~and~~ and $\phi_{n+1}(x)$. Referring to the curves A and B, we notice that, in consonance with (17), the ordinate of B corresponding to the farthest node of A on the x-axis (i.e the largest zero of ϕ_n) decreases with n. Hence we have

$$\frac{1}{\sqrt{\{2(n+1)+1-x^2_{1,n+1}\}}} < \frac{1}{\sqrt{\{2n+1-x^2_{1,n}\}}}$$

$$(x_{1,n+1} > x_{1,n} \text{ vide}(12))$$

Since only positive values of the square root are being considered, this leads, on simplification to the inequality

$$x^2_{1,n+1} - x^2_{1,n} < 2 \qquad (19)$$

The plausibility of (19) can be checked by using the zeros of as given in the table of Salzer for $n = 1(1)150$, taking values only up to 2D. But this is unnecessary since a proof of (19) can be given by using (15) or (15a).

Putting $r = 1$ for the largest zero we have, using the latter,

$$x_{1,n} = (2n)^{1/2} - 6^{-1/3} i_1 (2n)^{-1/6} + 0(n^{-1/6})$$

and similarly

$$x_{1,n+1} = (2n+2)^{1/2} - 6^{-1/3} i_1 (2n+2)^{-1/6} + 0\{(n+1)^{-1/6}\}$$

Squaring the two expressions, taking the difference, grouping the ~~terms~~ resulting terms suitably, we have, up to the order indicated,

$$x^2_{1,n+1} - x^2_{1,n} = 2 + \{(n+1)^{-1/3} - n^{-1/3}\}\{1 + 2^{-1/3} 6^{-2/3} i_1^2 - 2^{5/6} \cdot 6^{-1/3} i_1\}$$
$$+ \{(n+1)^{1/3} - n^{1/3}\}\{2^{3/2} - 2^{4/3} \cdot 6^{-1/3} i_1\} \qquad (20)$$

Using the expression (16) for the Airy's function, the second factor of the second term can be shown to be positive. Similarly the second factor of the third term can be shown to be negative. Since the first factors of the 2nd. and 3rd. terms are respectively $-$ ve, and $+$ ve, (20) leads to the desired inequality (19). A proof can also be given using (15), and noting that $\epsilon_{n+1} < \epsilon_n$.

6. Inequality relating to zeros of $\phi'_n(x)$ and $\phi'_{n+1}(x)$. Referring again to the curves A and B, using the inequality (5), the relation (17), and appealing to the correspondence principle in the form that the larger the value of n, the more nearly does the wave-mechanical probability

distribution function approximate to the classical case for the same energy, we shall make the intuitive assumption that the difference between the ordinates of the curves A and B at the point $x'_{0,n}$ decreases as n increases. This leads to

$$\phi^2_{n+1}(x'_{0,n+1}) - \frac{1}{\sqrt{(2n+3-x'^2_{0,n+1})}} < \phi^2_n(x'_{0,n}) - \frac{1}{\sqrt{(2n+1-x'^2_{0,n})}}$$

i.e

$$\phi^2_n(x'_{0,n}) - \phi^2_{n+1}(x'_{0,n+1}) > \frac{1}{\sqrt{(2n+1-x'^2_{0,n})}} - \frac{1}{\sqrt{(2n+3-x'^2_{0,n+1})}}$$

Hence if the R.H.S of the above be > 0, so is the L.H.S, and this is true in virtue of (5) ~~§~~. In other words, if

$$2n+3-x'^2_{0,n+1} > 2n+1-x'^2_{0,n} \qquad (x'_{0,n+1} > x'_{0,n} \quad \text{vide}(13a))$$

$$\text{or if} \quad x'^2_{0,n+1} - x'^2_{0,n} < 2 \qquad \text{.....} \quad (21)$$

then (5) holds. Thus (21) does not amount to a deduction from (5), but is offered here as a conjecture, in view of the fact that we have not been able to derive an explicit expression for $x'^2_{\nu,n}$ or $x'_{\nu,n}$, which corresponds to (15). A table has, however, been prepared to verify the correctness of (21), by finding the limits between which the $\phi'^2_{0,n}$'s lie for n = 1(1)11, 2D. For this purpose, we shall use the expressions for the first ten values of n given by Pauling & Wilson[10], and then obtain the corresponding polynomial equations $\phi'_n(x)=0$ using (9a). $H_{11}(x)$ is obtained from the recurrence relation (2), and $\phi'_{11}(x)=0$ is obtained ~~in the same manner as above~~ by using (9). Taking out the root $x = 0$, appearing ~~in~~ for even values of n, the actual calculations for n = 3,4 reduce to solution of quadratics, for n = 5,6 to cubics, for n = 7,8 to quartics, for n = 9,10 to quintics, and for n = 11 to a sextic in the squares of $x'_{0,n}$. The largest zeros of these eleven polynomial equations were determined by using Horner's method. ~~Form~~ For values of $n \geqslant 7$, the arithmetical work becomes rather complicated. For these cases, it was originally proposed to use the computer, but, in view of the labour involved in preparing the software for the programming, it was decided to continue with ~~Horner~~ Horner's method. In the table thus prepared, and shown below, the ~~values of~~ upper and lower limits of the values of $x'^2_{0,n}$ for $n \geqslant 3$ are ~~indicate~~ indicated, and it can be easily verified that the difference between the upper limit of $x'^2_{0,n+1}$, and the lower limit of $x'^2_{0,n}$ is less than 2, for all the cases in the table, thus verifying (21). An asymptotic ~~tendency for this difference~~

tendency for this difference to approach the limit 2 might also be noticed.

n	$x'^{2}_{0,n}$ = or	> or <
1	1	
2	2·50	
3	4·13	4·14
4	5·84	5·85
5	7·59	7·60
6	9·38	9·39
7	11·18	11·19
8	13·00	13·01
9	14·84	14·85
10	16·69	16·70
11	18·55	18·56

Table showing the limits
between which $x'^{2}_{0,n}$'s lie for $n = 1$ to 11.

Acknowledgment

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P.T.O

(7)

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To, 14/2/70

The Editor,
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Sir,

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Quantum Mechanical interpretation of an inequality concerning the Hermite orthogonal functions.

B. S. Madhavarao.

~~Abstract~~:

Abstract. As is well known the Hermite orthogonal functions are the eigen-functions of the quantum-mechanical problem of the linear harmonic oscillator. Using this, and the notions of the probability density for the quantum and the classical states based on the correspondence principle, ~~[illegible]~~ a known inequality relating to the Hermite functions has been employed to derive two new types of asymptotic inequalities. One of these relates to the largest zeros of the Hermite orthogonal function ~~polynomial~~, and the other to the largest zeroes of the derivative of this function ~~polynomial~~. The first inequality has been proved, and the second is presented ~~offered~~ here as a conjecture. A table ~~with~~ calculated for $n = 1(1)10$, 2D is given ~~offered~~ to show the plausibility of this conjecture.

1. Introduction.

The ~~general~~ normalised eigenfunctions of the linear harmonic oscillator are given by

$$\phi_n(x) = (\pi^{1/2} 2^n n!)^{-1/2} e^{-x^2/2} H_n(x) \qquad (1)$$

where $H_n(x)$ is the Hermite polynomial satisfying the recurrence relation

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \quad \underline{n} \geq 1, \qquad (2)$$

using which, and the relations $H_0(x) = 1$, and $H_1(x) = 2x$, one can derive ~~[illegible]~~ ~~$H_n(x)$~~ polynomial expressions for $H_n(x)$ for higher values of $\underline{n}$. Such values for $n = 1(1)10$ have been given by Pauling & Wilson[1], and we shall have occasion to use these later on. We can modify (1) by dropping the factor $\pi^{-1/4}$, ~~which [illegible]~~ and write the eigenfunctions as

$$\psi_n(x) = (2^n n!)^{-1/2} e^{-x^2/2} H_n(x). \qquad (1,a)$$

The inequality in question is that

$$|\psi_n(x)| \leq \max|\psi_0(x)| = \psi_0(0) = 1 \qquad (3)$$

proved by Otto Szasz[2], ~~and~~ thereby ~~[illegible]~~ confirming a conjecture of John Todd[3], and giving a refinement of the corresponding inequality of Erdelyi[4]. It might also be mentioned that the same result (3) has been derived by Jack Indritz[5], using ~~the same method as~~ a method identical with that of Szasz, in the form

$$\max|\phi_n(x)| \leq \max|\phi_0(x)| = \pi^{-1/4} \qquad (3,a)$$

2. ~~Equivalent~~ Alternative form of the inequality.

For the purposes of ~~[illegible]~~ the quantum mechanical interpretation, and derivation of the inequalities mentioned in the Abstract, We shall put ~~the~~ (3) in an alternative form, also given by Szasz[6]. Denoting the successive relative maxima of $|\phi_n(x)|$ as x decreases from $+\infty$ to 0 by

$$\mu_{0,n},\ \mu_{1,n},\ \mu_{2,n},\ \ldots \qquad (4)$$

it follows from a well-known theorem due to ~~Sonine~~ Sonin or Sonine-Polya[7], relating to orthogonal polynomials, that the successive relative maxima of $e^{-x^2/2}|H_n(x)|$ form an increasing sequence for $x \geq 0$[8] i.e the sequence (4) is decreasing, or

$$\mu_{r,n} > \mu_{r+1,n} \quad (r = 0, 1, \ldots) \qquad (4,a)$$

The inequality (3) therefore amounts to the statement that

$$\mu_{r,n} > \mu_{r,n+1}, \quad n \geq r \geq 0 \qquad (5)$$

Since we are ~~interested~~ concerned in this paper only with the highest relative maximum in (4), we shall use (5) in the form

$$\mu^2_{0,n} > \mu^2_{0,n+1} \qquad (5,a)$$

~~3. Quantum-mechanical interpretation considerations.~~

~~3. Quantum considerations.~~

~~3. Prof Probability distributions.~~

3. Zeros of $\phi_n(x)$ and $\phi_n'(x)$. - Before deriving the asymptotic inequalities relating to these zeros based on quantum mechanical considerations, we shall summarise below some of the well-known ~~results~~ relevant results[2]:

The zeros of $\phi_n(x)$ are, as is evident from (1) (2), the zeros of $H_n(x)$ and it is known that $H_n(x)$ has ~~all its~~ n zeros ~~real~~ all real and simple, say

$$x_{1,n} > x_{2,n} > \cdots x_{n-1,n} > x_{n,n}. \qquad (6)$$

$\phi_n(x)$ vanishes at these points and at $x = \pm\infty$, which might be denoted by $x_{0,n}$ and $x_{n+1,n}$ respectively. We also have the standard formulae for the derivative of $H_n(x)$ viz.

$$H_n'(x) = 2n H_{n-1}(x) \qquad (7)$$

$$H_n(x) = 2x H_{n-1}(x) - H'_{n-1}(x)$$

so that

$$H_n'(x) = 2x H_n(x) - H_{n+1}(x) \qquad (8)$$

From (1, a), we have

$$\phi_n'(x) = \frac{e^{-x^2/2}}{(\pi^{1/2} 2^n n!)^{1/2}} (H_n' - x H_n)$$

so that the zeros of $\phi_n'(x)$ are given by the ~~zeros~~ zeros of

$$H_n'(x) - x H_n(x) = 0 \qquad (9)$$

or, using (8), ~~those~~ of $x H_n(x) - H_{n+1}(x) = 0$ (9, a)

By Rolle's theorem, ~~it follows that~~ between any two roots of $\phi_n(x) = 0$ lies at least ~~one root of (9) or (9, a) i.e~~ of $\phi_n'(x) = 0$, so that as ~~seen~~ is evident from (9) or (9, a), $\phi_n'(x) = 0$ has $n+1$ roots, say

$$x'_{0,n} > x'_{1,n} > \cdots > x'_{n,n} \qquad (10)$$

and in addition $\phi_n'(x)$ vanishes at $x = \pm\infty$, which might be denoted by $x'_{-1,n}$ & $x'_{n+1,n}$ respectively. Thus it follows that

$$x'_{0,n} > x_{1,n} > x'_{1,n} > x_{2,n} > \dots > x_{n,n} > x'_{n,n} \qquad (11)$$

and similarly, $x'_{0,n+1} > x_{1,n+1} > x'_{1,n+1} > x_{2,n+1} > \dots$ (11, a)

Again (1.7) shows that the roots of $H_n(x)$ are the roots of H'_{n+1} and are interlaced with the roots of H_{n+1}, thus

$$x_{1,n+1} > x_{1,n} > x_{2,n+1} > x_{2,n} \dots > x_{n,n+1} > x_{n,n} > x_{n+1,n+1} \qquad (12)$$

Hence,

$$x'_{0,n+1} > x_{1,n+1} > x_{1,n} \text{ and } x'_{0,n} > x_{1,n} > x'_{1,n} \qquad (13)$$

Finally, we have the results

$$x'_{0,n} < \sqrt{(2n+1)}, \quad -\sqrt{(2n+1)} < x'_{n,n} \qquad (14)$$

and that the largest zero of ϕ'_{n+1} is $> x'_{0,n}$ ie

$$x'_{0,n+1} > x'_{0,n} > x'_{1,n+1} > x'_{1,n} > \dots > x'_{n+1,n+1} \qquad (15)$$

The inequalities (6), (10), (11) – (15) give all the information re. the location of the roots of ϕ_n, ϕ_{n+1}, ϕ'_n, ϕ'_{n+1} and how they are relatively interlaced. ~~The results remain the same if the ψ's be replaced by the φ's~~.

As regards the actual expressions for the roots of $\phi_n(x)$ and $\phi'_n(x)$, it might be noted that although the roots of $\phi_n(x)$ ~~being~~ are the same as those of $H_n(x)$, and that $H_n(x) = 0$ ~~giving~~ gives polynomial equations in $x^2_{r,n}$ $(r = 1, \dots n)$ (taking out the root $x = 0$ for the case $n =$ odd), there are no direct expressions for $x^2_{r,n}$ itself to be found in the literature. ~~There is, however, a remarkable expression for $x_{r,n}$ itself given by~~ There are, however, some remarkable results[9] relating to ~~the inequalities the largest zero x_{1n}~~, as well as to $x_{r,n}$. ~~for fixed values of r as $n \to \infty$~~ Thus one has the inequality

$$x_{r,n} < (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6} i_r \qquad (16)$$

where $i_1 < i_2 < i_3 < \dots$ $(i_1 > 0)$ are the real zeros of Airy's function[9] $A(x)$.

Further, for a fixed r,

$$x_{rn} = (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6}\{i_r + \epsilon_n\} \qquad (17)$$

where $\lim_{n\to\infty} \epsilon_n = 0$. It follows from (16) and (17) that the constant i_r is the best possible if r is fixed, and n arbitrary, and also that $(2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6} i_1$ is an upper bound for the zeros of $H_n(x)$, and that the constant

$$6^{-1/3} i_1 = 1.85575\dots \qquad (18)$$

cannot be replaced by a smaller one. An alternative form for (17) is given by

$$x_{r,n} = (2n)^{1/2} - 6^{-1/3} i_r (2n)^{-1/6} + O(n^{-1/6}), \quad n \to \infty \qquad (17, a)$$

From the computational point of view, we might mention the tabulation by H. E. Salzer, ..[10] of the largest zeros of $H_n(x)$ for $n = 1\ (13)\ 15D$, $n = (14-16)\ 14D$, and $n = (17-20)\ 13D$., and also the asymptotic formulae for the Hermite polynomials by H. E. Salzer.[11]

Regarding the zeros of $\phi'_n(x)$, no information either regarding to

expressions for the zeros or any computational approach is available (4)
in the literature except, perhaps, for the ~~very general~~ general theorem[12] that if the system $\phi_n'(x)$ of the derivatives of an orthogonal system $\phi_n(x)$ be also orthogonal, then $\phi_n(x)$ must be the system of Jacobi, Laguerre or Hermite polynomials, and the generalisation of this theorem[13] for the system $\phi_n^{(r)}(x)$ of the r[15] derivatives.

4. ## Quantum-theoretic considerations.

We use the physical interpretations of ϕ_n based on the notion of ~~probability~~ probability i.e. that $\phi_n^2\,dx$ defines the probability of occurrence of the particle in the element of distance dx at the point x., or, in other words that ϕ_n^2 is the probability density at x. The ~~corresponding~~ probability distribution function for the ~~classical~~ corresponding linear harmonic oscillator ~~is given by~~ (of frequency ν_0 and total

$$\cancel{R\,dx = \frac{dx}{\pi\sqrt{(2n+1-x^2)}}}$$

energy given by the eigen-value $E_n = h\nu_0(n+\tfrac{1}{2})$ corresponding to the eigenfunction ϕ_n) is given by[14]

$$P\,dx = \frac{dx}{\pi\sqrt{(2n+1-x^2)}} \qquad \text{--- (19)}$$

~~on dropping the factor $\frac{1}{\pi}$ in~~

Equation (19) shows that, for the classical oscillator, the probability density increases from $1/\pi\sqrt{2n+1}$ at $x = 0$ to ∞ at $x = x_0 = \pm\sqrt{(2n+1)}$. ~~Also~~ --- (20)

Also from the Schrödinger equation

$$\frac{d^2\phi}{dx^2} + (2n+1-x^2)\,\phi = 0 \qquad (21)$$

we have $\frac{d^2\phi}{dx^2} = 0$ for $x = x_0$ showing that this constitutes a point of inflexion. For values of $|x| > |x_0|$, ϕ decreases continuously without without exhibiting any nodes.

Plotting ϕ_n^2 against x (for ~~several~~ different values of n) we get the ~~curves~~ probability distribution curves (denoted by A (~~Fig~~) as in the Figures shown) for the quantum cases, which indicate the statistical interpretation of the Schrödinger equation in consonance with the principle of indeterminacy. Using equation (19) the probability distribution curves, denoted by ~~the dotted lines~~ B (dotted curves) are also shown ~~indicated~~ in the same figures. ~~Such curves are to be found in most of the standard text-books on quantum mechanics for the successive values of $n = 0(1)4$. We have extended them here to the cases $n = 5, \ldots$ using the table of Hermite functions given by J. B. Russell~~

The curves indicated are drawn for the cases $n = 0\,(1)\,4$ and are plotted for ~~both~~ $+^{ve}$ ~~and $-^{ve}$~~ values of x. ~~(obviously symmetrical about the ϕ^2-axis)~~. Since similar curves, sym. w.r.t the ϕ^2-axis, may be plotted for $-^{ve}$ values of x, and since the ϕ's are normalised, we have the result that the total area under the curve for $-\infty \leq x \leq \infty$ is equal to 1. These curves illustrate clearly the results regarding the location of the roots of ϕ_n, ϕ_{n+1}, ϕ^{q}_{n+1} & ϕ'_{n+1} as given by equations (6), (10), (11)-(15), and (20). Similarly the inequalities (5) and (5, a) are also illustrated in the figures. In particular, we might ~~st~~ interpret (5, a) by saying that the maximum probability density decreases ~~with~~ with ~~the~~ increase of quantum number $\underline{n}$. This is in consonance with the principle of correspondence, since the corresponding classical probability density given by (19) decreases with $\underline{n}$. This argument is, of course, no substitute for the rigorous mathematical deductions of (5) and hence of (5, a).

5. Asymptotic ~~I~~nequalities between zeros of $\phi_n(x)$ and $\phi'_n(x)$.

Referring to the figure ~~giving~~ showing the probability distribution curves A and B, we notice that, in consonance with (19), the ~~ordinate~~ ordinate of the curve B corresponding to the farthest node of the curve A on the x-axis (ie. ~~corresponding to~~ the largest zero of $\phi_n(x)$) decreases as $\underline{n}$ increases. Hence we have

$$\frac{1}{\sqrt{2(n+1)+1-x_{1,n+1}^2}} < \frac{1}{\sqrt{2n+1-x_{1,n}^2}}, \qquad (x_{1,n+1} > x_{1,n}, \text{ vide } \cancel{\text{supra}}\,(12))$$

Since only $+^{ve}$ values of the square roots are concerned, this leads on simplification to the results

$$x_{1,n+1}^2 - x_{1,n}^2 < 2, \tag{22}$$

an asymptotic inequality for the squares of the zeroes of $\phi_n(x)$.

Again, using the Correspondence principle in the form that the larger the value of $\underline{n}$ the more nearly does the wave-mechanical probability distribution function approximate to the classical ~~exp~~ expression for the same energy, we shall make the intuitive assumption that the difference in the ordinates of the curves A and B ~~at the point~~ at the point ~~of~~ of the maximum probability density (ie the point $x'_{0,n}$) decreases as $\underline{n}$ increases. ~~Since It must be emphasised that this is not a rigorous deduction, but~~ Hence we have

$$\phi_{n+1}^2(x'_{0,n+1}) - \frac{1}{\sqrt{2(n+1)+1-x_{0,n+1}'^2}} < \phi_n^2(x'_{0,n}) - \frac{1}{\sqrt{2n+1-x_{0,n}'^2}}$$

$$\text{ie, } \phi_n^2(x'_{0,n}) - \phi_{n+1}^2(x'_{0,n+1}) > \frac{1}{\sqrt{2n+1-x_{0,n}'^2}} - \frac{1}{\sqrt{2n+3-x_{0,n+1}'^2}}.$$

Hence if the R.H.S. of the above inequality be >0, so is the L.H.S., and this is rigorously true in virtue of (5, a). In other words, if

$$2n+3-x_{0,n+1}'^2 > 2n+1-x_{0,n}'^2 \qquad (x'_{0,n+1} > x'_{0,n}, \text{ vide } (15))$$

$$\text{or if } \cancel{x_{0,n}'^2 \sim x_{0,n+1}'^2}\ x_{0,n+1}'^2 - x_{0,n}'^2 < 2 \tag{23}$$

then (5, a) holds. Thus (23) ~~amounts to a conjecture only~~ does not amount to a deduction from (5, a), but is merely a conjecture.

We can verify the inequality (22) by using the values of the zeros of the Hermite polynomials for $n = 1(1)20$ only up to 2D, but this is unnecessary since a proof can be given using the exp$^{\underline{n}}$ for ~~x~~ these zeros given by (17, a). Putting ~~zero~~ $r = 1$ for the largest zero, we have $x_{1,n} = (2n)^{1/2} - 6^{-1/3} i_1 (2n)^{-1/6} + O(n^{-1/6})$, ~~Hence~~ and similarly $x_{1,n+1} = (2n+2)^{1/2} - 6^{-1/3} i_1 (2n+2)^{-1/6} + 0\{(n+1)^{-1/6}\}$

Squaring these expressions, taking the difference, and grouping the terms we get

$$x_{1,n+1}^2 - x_{1,n}^2 = 2 + 6^{-2/3} i_1^2 \{(2n+2)^{-1/3} - (2n)^{-1/3}\} + [0\{(n+1)^{-1/3}\} - 0(n^{-1/3})]$$
$$-2.6^{-1/3} i_1 \{(2n+2)^{1/3} - (2n)^{1/3}\}$$
$$+2[(2n+2)^{1/2} 0\{(n+1)^{-1/6}\} - (2n)^{1/2} 0(n^{-1/6})]$$
$$-2.6^{-1/3} i_1 [(2n+2)^{-1/6} 0\{(n+1)^{-1/6}\} - (2n)^{-1/6} 0(n^{-1/6})] \quad \text{--}(17,b)$$

Taking the terms up to the order indicated, and grouping the second, third and sixth terms, we get.

$$\{(n+1)^{-1/3} - n^{-1/3}\}\{1 + 2^{-1/3} 6^{-2/3} i_1^2 - 2^{5/6}.6^{-1/3} i_1\};$$

and using expn (18) for the Airy's function, we can easily show that the second factor is $+^{ve}$ and the first factor being $-^{ve}$, the product is $-^{ve}$. Similarly grouping the 4th and 5th terms we get

$$\{(n+1)^{1/3} - n^{1/3}\}\{2^{3/2} - 2^{4/3}.6^{-1/3} i_1\};$$

Again using (18), the second factor is shown to be $+^{ve}$ while the first factor is $-^{ve}$ & hence the product $-^{ve}$. Hence (17.b) leads to the relation (22). A proof can also be given using (17), and noting that $\epsilon_{n+1} < \epsilon_n$.

Regarding the proof of the conjecture (23), there are no expressions available in the literature, corresponding to (17) or (17, a). The differential equation satisfied by $\phi_n'(x)$ can be easily derived, but the further application of the method ~~followed~~ used by Szego to obtain (17) meets with great difficulties. We shall therefore content ourselves in the present paper by preparing a table for verification of (23), by finding the ~~limits~~ limits between which the $x_{0,n}'$ lie for $n = 1\,(1)\,10$, 2D. For this purpose we shall use the expressions for $H_n(x)$ for these ten values of n, ~~and~~ obtain the corresponding polynomial equations $\phi_n'(x) = 0$ given by (9) or alternatively (9, a). We shall find the largest zeros of these polynomials by using Horner's method. ~~The table is appended below.~~ The actual calculations ~~become~~ for $n = 3$ & 4 ~~up to n = 4~~ reduce to soln of quadratics, for 5, 6 to cubics, 7 & 8 to quartics and 9 & 10 to quintics in $x_{0,n}'^2$, and calculations up to 2D by Horner's process ~~are do not call for~~ ~~Computer~~

n	$x_{0,n}'^2 =$ or $><$	
1	1	
2	2.50	
3	4.13	4.14
4	5.84	5.85
5	7.59	7.60
6	9.38	9.39
7	11.18	11.19
8	13.00	13.01
9	14.84	14.85
10	16.69	16.70
11	18.55	18.56

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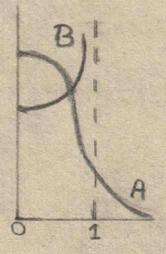
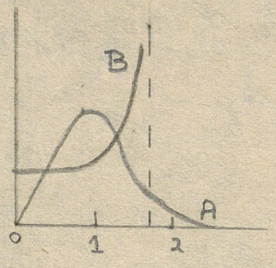
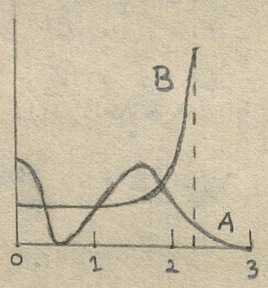


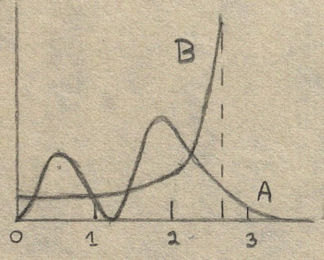
$n = 0$



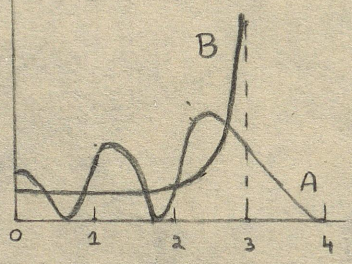
$n = 1$



$n = 2$



$n = 3$



$n = 4$

$F_n = F_{n-2} + F_{n-1}$ $\quad F_1 = F_2 = 1$

$F_3 = 1 + 1 = 2$

$F_4 = 1 + 2 = 3$

$F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $F_{11} = 89$

$F_{12} = 144$