

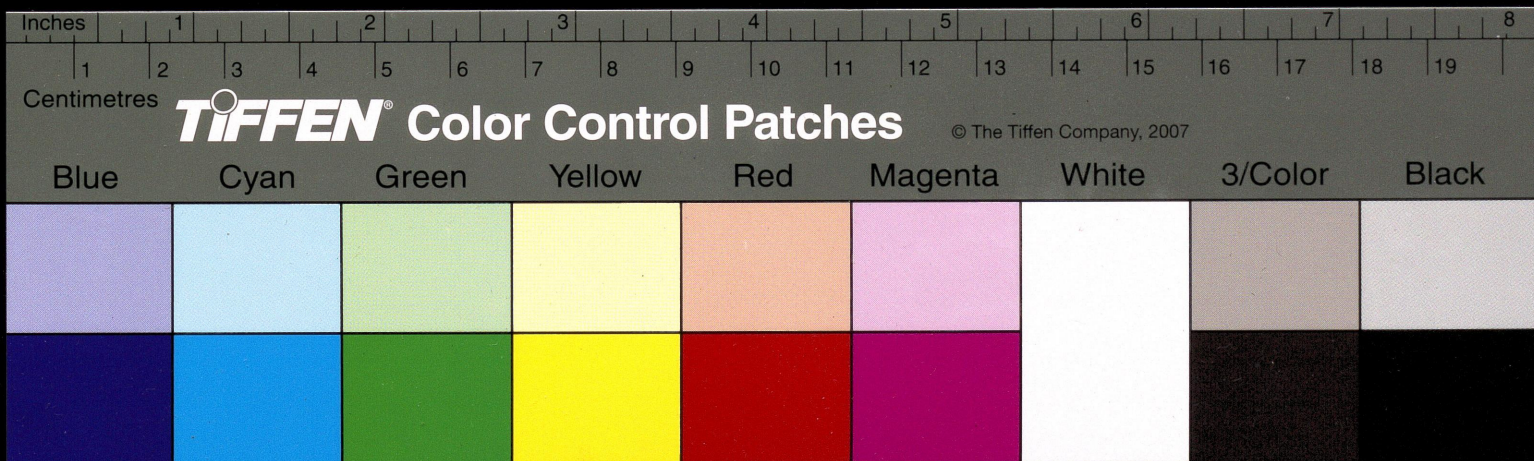
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MIXED POLAR THEOREMS ON PLANE CUBICS.

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**MIXED POLAR THEOREMS ON
PLANE CUBICS.**

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§ 1. INTRODUCTION.

I have recently given¹ a mixed polar theorem relating to the pencil of four tangents that can be drawn to a non-singular plane cubic from a point on the curve itself. In the course of the investigations leading on to this theorem, I have been able to derive a number of other subsidiary theorems dealing with mixed polars and poloconics which, although not directly connected with the mixed polar theorem, have an intrinsic interest. These theorems have been given in §2 of this paper. Further theorems which follow from a direct application of the mixed polar property with the aid of pure geometric and symbolic methods have next been given in §3. It is shown in this section that some of the theorems given by Rey. J. Pastor² are immediate geometrical consequences of the mixed polar theorem.

§ 2. THEOREMS ON POLO-CONICS AND MIXED POLARS.

THEOREM 1.—*The mixed polar of any two points P, Q in the plane is the polar w.r.t. the poloconic of PQ of the point of intersection of the polar lines of P and Q.*

Let the cubic be $a_x^3 \equiv b_x^3 = 0$ and the points P and Q be y and z and the line PQ, $u=0$. The poloconic of PQ is given by

$$\ominus = (abu)^2 a_x b_x = 0$$

which can also be written as

$$\ominus = \theta_x^2 \phi_u^2 = 0 \quad \dots \quad (1)$$

Let the polar lines of P and Q meet at the point t . The polar of this point w.r.t. (1) is given by

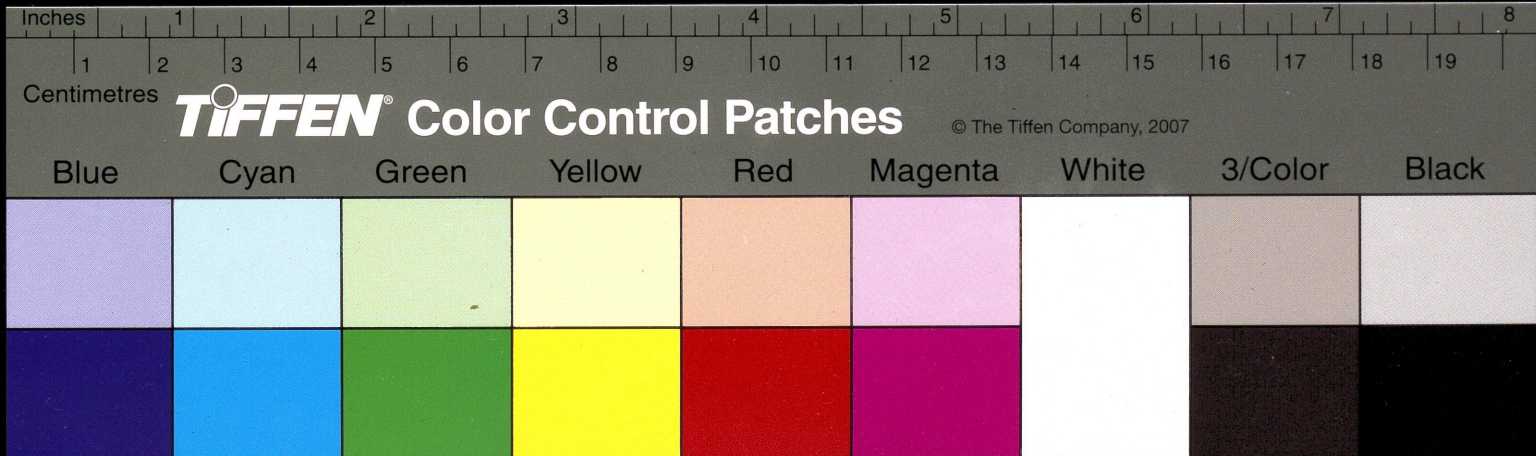
$$\theta_x \theta_t \phi_u^2 = 0.$$

We have also the relations

$$a_t a_y^2 = 0 = a_t a_z^2 \quad \dots \quad (2)$$

¹ *Proc. Ind. Acad. Sci.*, Vol. 1, No. 6, (1934), p. 367.

² *Atti. Accad. naz. Lincei, Rend.*, VI, S. 20 (1934), pp. 459-62.



Operating with $\left(a \frac{\partial}{\partial \theta} \right) \left(b \frac{\partial}{\partial \theta} \right) \left((ab) \frac{\partial}{\partial \phi} \right)^2$ on $\theta_x \theta_t \phi_u^2$

we get $(abu)^2 (a_x b_t + b_x a_t) = 0$

which is equivalent to

$$(a_y b_z - a_z b_y)^2 (a_x b_t + b_x a_t) = 0$$

since y and z are points on $u=0$. Expanding the left-hand side of this last equation and using (2) we derive

$$a_y a_z b_y b_z (a_x b_t + b_x a_t) = 0$$

and remembering that a and b are equivalent symbols this gives

$$(a_x a_y a_z) (b_t b_y b_z) = 0$$

i.e., the equation required is $a_x a_y a_z = 0$ which is the mixed polar of P and Q.

COR.—*The mixed polar of P, Q touches the poloconic of PQ if P coincides with Q at a point of the cubic.*

Let the mixed polar of P, Q meet $u=0$ at R; the points P, Q, R then form a degenerate apolar triangle of the cubic. The polar lines of P, Q, R form a triangle circumscribed to the poloconic and the three lines joining these points of contact are the mixed polars of the three pairs P, Q; Q, R and R, P. Hence we have

THEOREM 2.—*The triangle formed by the mixed polars by pairs of three collinear points forming a degenerate apolar triangle is inscribed in the poloconic of the line.*

The above theorems are very much simplified and can be made to yield further interesting results when P and Q are on the cubic itself. The point R will then be the third intersection of the line PQ ($u=0$) with the cubic and Theorem 2 transforms itself into

THEOREM 3.—*The triangle formed by the mixed polars by pairs of the three points of intersection of a cubic by a line is inscribed in the poloconic of the line.*

As a simple corollary we can deduce the condition in order that the mixed polar of P, Q may be the tangent to the cubic at R. This is obviously true if P, Q and R coincide in which case the mixed polar becomes the inflexional tangent. An alternative case arises when the poloconic of the line PQR degenerates into a pair of st. lines, for, the point of intersection of the polar lines of P and Q will be the double point of this degenerate poloconic and by Theorem 1, the mixed polar of P, Q coincides with the line joining this double point to R, i.e., the tangent at R. Hence the

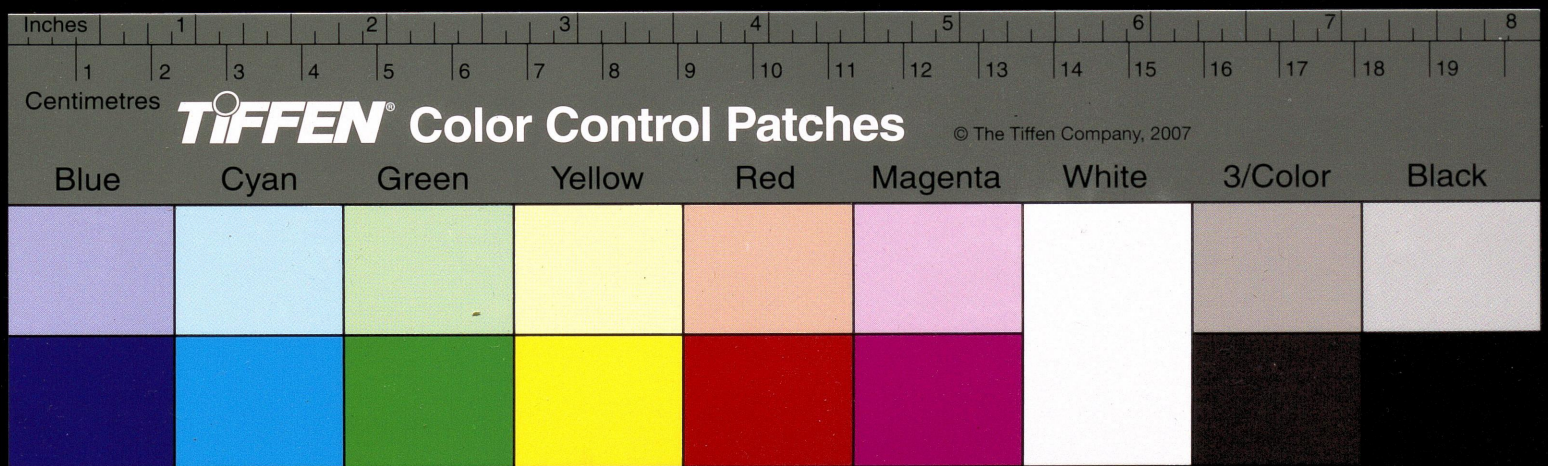
COR.—*The mixed polar of P and Q is the tangent at R either when P, Q, R coincide or when the line PQR has a degenerate*

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poloconic. In the latter case the mixed polar of any of the three pairs is the tangent at the residual point.

We can now proceed further and investigate how the mixed polars of the three pairs of points are related to the points P', Q', R' which are the other intersections with $u=0$ of the polar conics of P, Q, R respectively. Referring to the adjoining diagram³, let the tangents at P, Q, R to the cubic form the

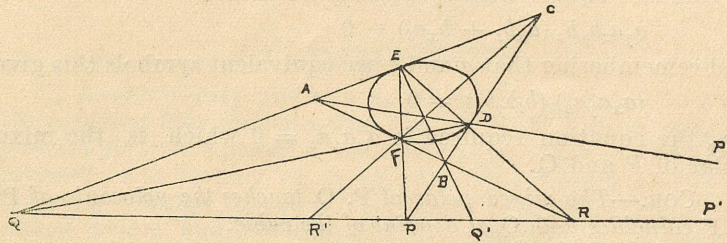


FIG. 1.

triangle ABC which is circumscribed to the poloconic of $u=0$. If D, E, F are the points of contact of these tangents with the poloconic it follows by Theorem 3 that DE, EF, FD pass through R, P, Q respectively and are the mixed polars of $P, Q; Q, R$ and R, P . The mixed polar of Q, P passes through R or in other words, the polar of Q w.r.t. the polar conic of P passes through R , which means that Q and R are conjugate points for the polar conic of P , i.e., (PP', QR) is a harmonic range. The points P', Q', R' therefore determine the cubic covariant of the binary cubic form given by P, Q, R , the Hessian covariant being the intersections of u with the poloconic of u . Hence using Clebsch's principle of translation the points P', Q', R' are given as the intersections of u with the curve

$$Q = (abu)^2(cau)c_x^2 b_x = 0.$$

But from a property of this curve⁴ the points $P, P'; Q, Q'$ and R, R' are conjugate points relative to the poloconic of u . Therefore AD, BE, CF pass through P', Q' and R' respectively. Consider further the polar of E w.r.t. the polar conic of Q . Since the polar of R w.r.t. this conic, i.e., the mixed polar of Q, R is PE , it follows that the polar of E w.r.t. the same conic passes through R . EQ being the tangent at Q to the cubic is also the tangent at Q to the polar conic of Q . Hence the polar of E w.r.t. this conic passes through Q , i.e., is the line QR or QQ' showing that EQ' is tangent at Q' to the polar conic of Q . This gives us

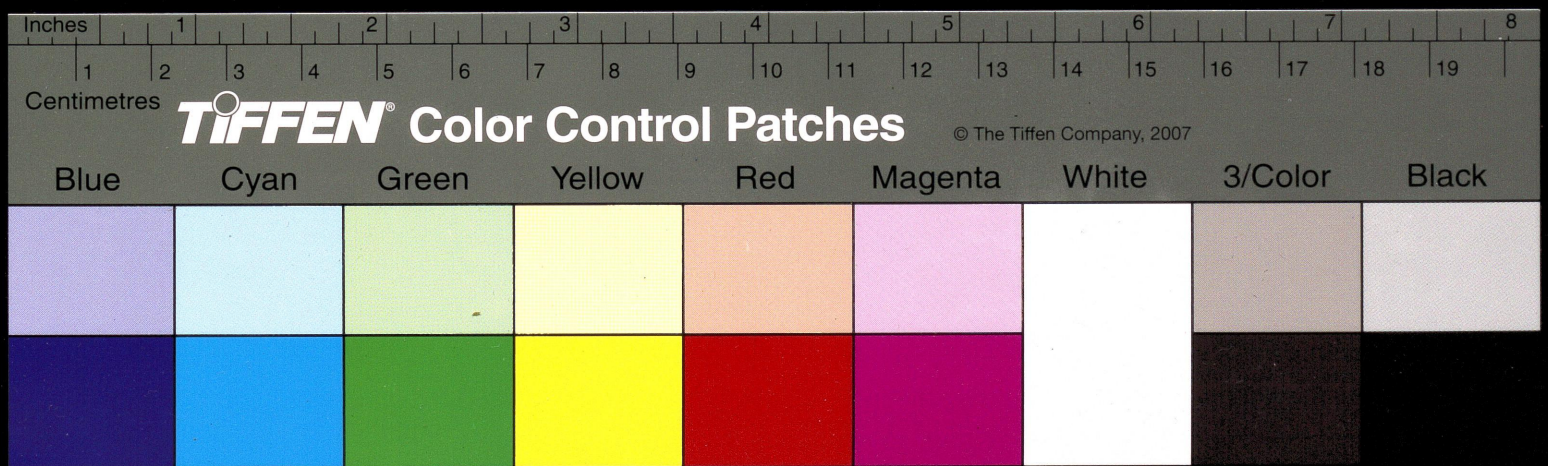
³ This diagram is the same as given in Salmon, H.P.C. (3rd Edition), p. 157

⁴ See Clebsch, Lindemann and Benoist, *Leçons*, t. 2 (1880), p. 279.

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THEOREM 4.—*If a line u meets a cubic in P, Q, R the vertices of the triangle formed by the mixed polars by pairs of these three points are the poles of u w.r.t. the polar conics of P, Q, R .*

An application of the well-known theorem⁵ that the polar conic of a point x is harmonically inscribed to the poloconic of a line u when x lies on u , gives a further interesting result. As mentioned above, P, P' are conjugate points w.r.t. the poloconic of u and hence the point of intersection of the polars of P, P' w.r.t. the same conic forms with P, P' a self-conjugate triangle for the poloconic. From the above theorem of Clebsch this must be a triangle inscribed in the polar conic of P and P, P' being points on this polar conic, the third vertex of this triangle should also lie on it. Hence the

THEOREM 5.—*The polars of P, P', Q, Q' and R, R' w.r.t. the poloconic of u meet on the polar conics of P, Q and R respectively.*

§ 3. THEOREMS ON SALMON PENCILS.

I call the pencil of four tangents that can be drawn from a point P on a cubic to the curve itself as the "Salmon pencil of P " and shall proceed to deduce some theorems relating to Salmon pencils by the use of the mixed polar theorem referred to in §1 which can be stated as follows:—

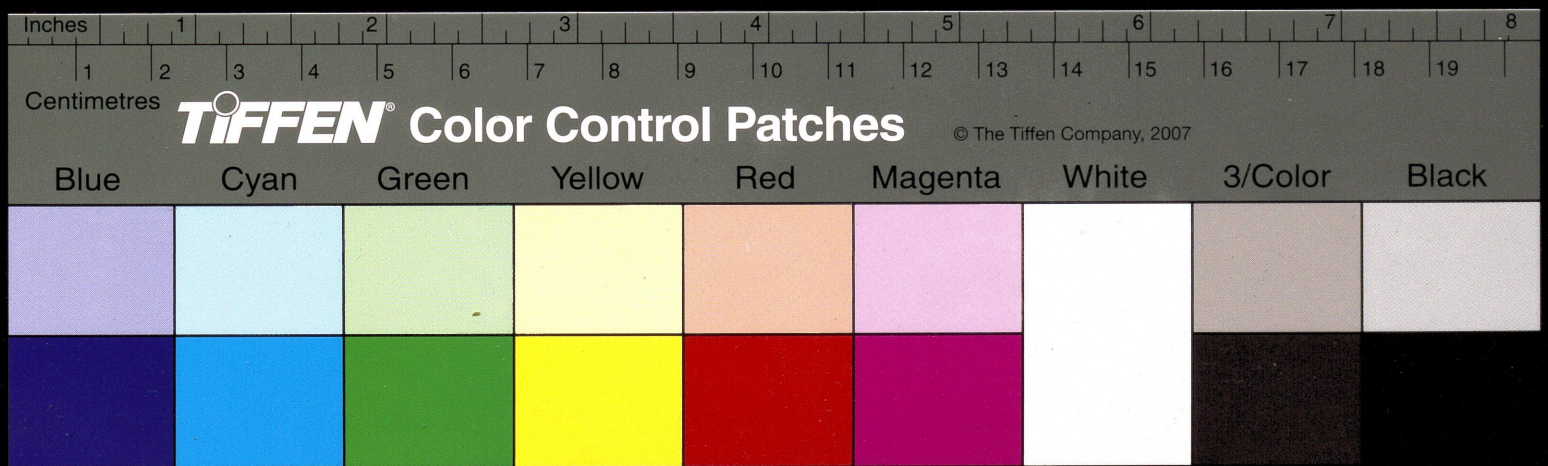
MIXED POLAR THEOREM.—*The polar line of any point P_1 in the plane w.r.t. the Salmon pencil of a point R on the cubic is the same as the mixed polar of the points P and Q in which the line P_1R meets the cubic again.*

If the points P and Q coincide at P_1 the point R will be the tangential of P and the mixed polar of P, Q becomes the line PR itself and hence passes through P_1 . Given the point P_1 there are however six points like R , viz., the tangentials of the points of contact of tangents from P_1 to the cubic, or the intersections of the cubic with the satellite conic of P_1 . Hence we have

THEOREM 6.—*The polar lines of any point in the plane w.r.t. the Salmon pencils of the six points of intersection of the cubic with the satellite conic of the given point are concurrent at the given point itself.*

If the point P_1 be on the cubic itself the polar line of P_1 w.r.t. the Salmon pencil of any point R on the cubic is the same as the mixed polar of P_1 and the point P which is the third point of intersection of P_1R with the cubic. As a particular case let P_1 be a point of inflexion on the cubic; the mixed polar of P_1 and P is the same as the polar of P w.r.t. the polar conic of P_1 which now degenerates into the inflexion tangent at P_1 and

⁵ Clebsch, Lindemann and Benoist, *ibid.*, p. 237.



the harmonic polar. Hence the mixed polar passes through the intersection of these lines and this gives *Rey Pastor's first theorem*. The polar lines of an inflexion point P_1 of C^3 w.r.t. the Salmon pencils of points on C^3 all pass through a fixed point which is the intersection of the inflexional tangent at P_1 and the harmonic polar of P_1 .

If P_1 be any fixed point of the curve other than a point of inflexion we can consider the problem of finding the envelope of the polar lines of P_1 w.r.t. the Salmon pencils of the points on C^3 . By the mixed polar theorem this reduces to the envelope of the mixed polars of P_1 and the different points on the cubic. If P_1 be denoted by y and any point on the cubic by ξ , the envelope mentioned is the envelope of the line

$$a_x a_y a_\xi = 0$$

with the condition $a_\xi = 0$

and hence is in general a curve of the third class. Hence we have

THEOREM 7.—*The envelope of the polar lines of a given point on C^3 with respect to the Salmon pencils of the different points on the cubic is in general a curve of the third class.*

I hope to return in a further paper to an investigation of the properties of this class cubic.

Let us now consider two given points P_2 and P_1 in the plane and determine the points on C_3 such that the polar lines of P_1 w.r.t. the Salmon pencils of these points are all concurrent at P_2 . Let P_1 and P_2 be denoted by y and z and any point on the cubic by ξ . The Salmon pencil of ξ is represented by the quartic

$$p_x^4 = 4a_x^3 b_\xi^2 b_x - 3a_x^2 a_\xi b_x^2 b_\xi = 0 \quad \dots \quad (3)$$

and the polar line of y w.r.t. (3) can be found to be

$$a_y^3 b_x b_\xi^2 + 3a_y^2 a_x b_y b_\xi^2 - 3a_x a_y b_y^2 a_\xi b_\xi = 0 \quad \dots \quad (4)$$

If this passes through ξ we have

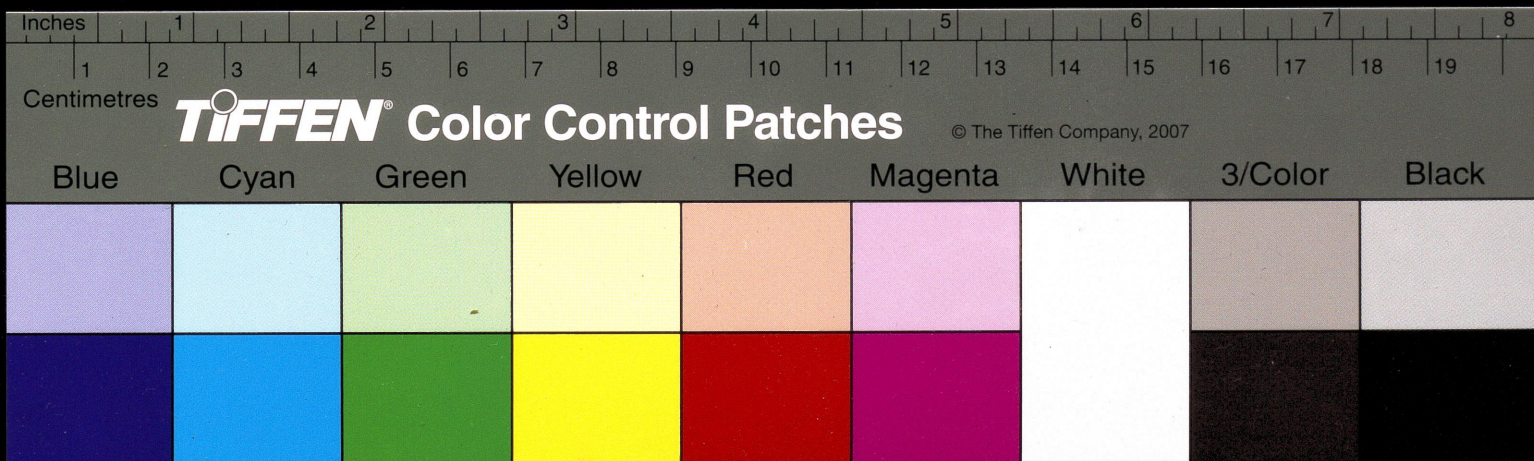
$$a_y^3 b_x b_\xi^2 + 3a_y^2 a_x b_y b_\xi^2 - 3a_x a_y b_y^2 a_\xi b_\xi = 0 \quad \dots \quad (5)$$

which shows that the required points are the intersections of the cubic with the conic

$$a_y^3 b_x b_x^2 + 3a_y^2 a_x b_y b_x^2 - 3a_x a_y b_y^2 a_x b_x = 0 \quad \dots \quad (6)$$

Hence the number of points sought for is six and we can state

THEOREM 8.—*Given two points P_1 and P_2 in the plane, there are six points on C^3 such that the polar lines of P_1 w.r.t. Salmon pencils of these six points all concur at P_2 .*



Reserving for a future occasion an investigation of the nature of the conic given by (6), I proceed to consider some particular cases. Let P_1 lie on the cubic or $a_y^3=0$. In this case (6) reduces to

$$a_y^2 a_z b_y b_x^2 - a_z a_y a_x b_y^2 b_x = 0 \dots \dots \dots (7)$$

which is of the form

$$\lambda C_y - P_{yz} P_y = 0$$

C_y being the polar conic of y , P_y the polar line of y and P_{yz} the mixed polar of y and z . Thus (7) passes through the intersections of C_y and P_y , i.e., it touches the cubic at y . This gives

THEOREM 9.—*There are four points on the cubic such that the polar lines of a point y on the curve w.r.t. the Salmon pencils of these four points concur at a point z .*

Specialising still further let us choose z such that $a_y^2 a_z = 0$ in addition to $a_y^3 = 0$. This means that z lies on the tangent at y . We might therefore take z to be the tangential of y . In this case (7) still further reduces to

$$(a_x a_y a_z) (b_y^2 b_x) = 0$$

i.e., $P_{yz} P_y = 0$

P_y meets the cubic in P_1 , P_1 and P_{yz} being mixed polar of P_1 and its tangential P_2 also passes through P_1 . Hence the six points of intersection of the conic (6) with C^3 reduce to P_1, P_1, P_1, P_2 and two other points P_1', P_2'' . The polar lines of P_1 w.r.t. Salmon pencils of P_1', P_1'' pass through P_2 . Using the mixed polar theorem these polar lines are the mixed polars of P_1, P_1' and P_1, P_1'' . Hence the

THEOREM 10.—*If P_1 be a point on the cubic and P_2 its tangential and if the mixed polar of P_1, P_2 meets the cubic in P, P_1', P_1'' , the mixed polars of P_1, P_1' and P_1, P_1'' meet at P_2 .*

Another special case of a different nature can be obtained by recalling the corollary to Theorem 3. Let P, Q, R be three collinear points on a cubic such that the poloconic of PQR degenerates into a pair of st. lines through P_2 . By a well-known theorem P_2 is on the Hessian and PQR is a tangent to the Cayleyan. The corollary referred to shows that the tangents at P, Q, R to the cubic pass through P_2 . If P', Q', R' be the points of contact of the other three tangents from P_2 to the cubic, P', Q', R' also lie on a line and the polar conic of P_2 is the pair of st. lines PQR , $P'Q'R'$ meet at P_1 which is also on the Hessian and is the conjugate pole of P_2 . Hence by corollary to Theorem 3, the polar lines of P_1 w.r.t. the Salmon pencils of these six points P, Q, R, P', Q', R' all concur at P_2 . This gives *Rey Pastor's second theorem*. To every point P_1 of the Hessian there correspond six Salmon pencils such that the

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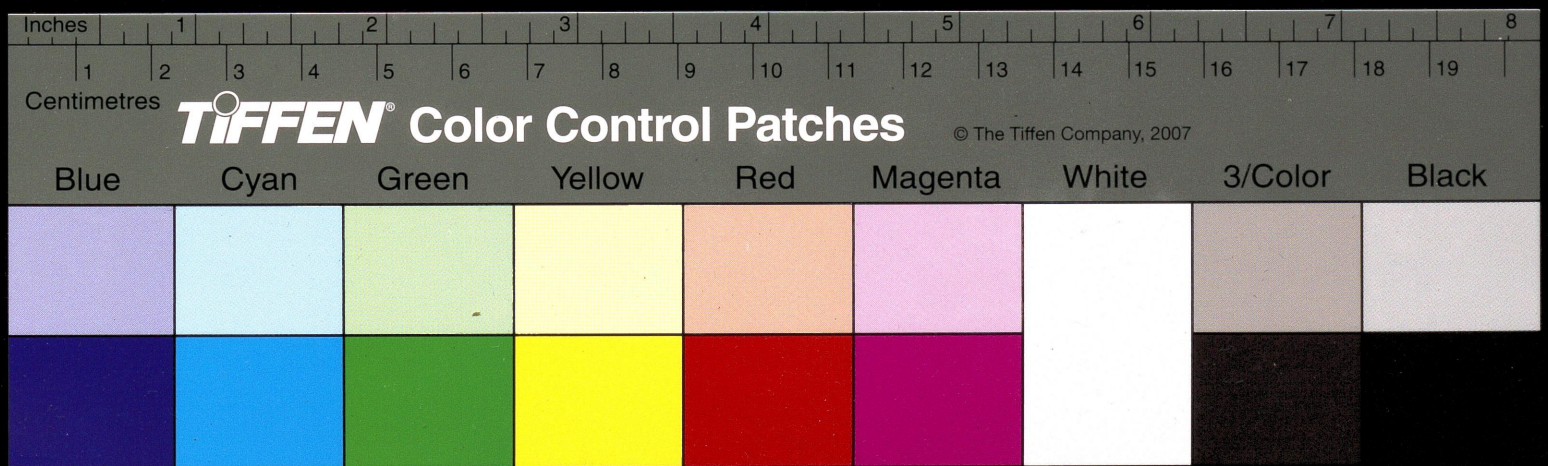
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polar lines of P_1 w.r.t. these pencils are concurrent at P_2 the Steinerian conjugate of P_1 .

This is also immediately evident from equation (6). For, if y and z be conjugate poles on the Hessian we have

$$a_y^2 a_z = 0$$

since polar conic of z passes through y and the equation $a_x a_y a_z = 0$ identically satisfied, i.e.,

$$\left. \begin{aligned} a_y a_z a_1 &= 0 \\ a_y a_z a_2 &= 0 \\ a_y a_z a_3 &= 0 \end{aligned} \right\}$$

Hence equation (6) reduces to

$$b_x^2 b_z = 0$$

which is the polar conic of $P_2(z)$ which immediately proves the above theorem.

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