Upper and lower bounds for the area of a triangle

by

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1. Introduction — ~~Various~~ Several inequalities for the area Δ (or its square Δ^2) of any plane triangle have been given by ~~Gerretsen (1)~~, Beatty (1), and Frucht (2), using ~~different methods various methods~~ various methods. We first of all indicate briefly that ~~all~~ these ~~inequalities~~ can be derived by a uniform method based on the inequalities related to the elementary symmetric functions of a set of positive numbers given by Hardy-Littlewood-Pólya (3), and a slight generalisation of the same.

We next set up a new type of inequalities which reduce to equalities in the case of right angled triangles, just as the inequalities of Beatty and Frucht reduce to equalities in the cases of equilateral and isosceles triangles respectively. A comparison is also made of this new type with those given in (1) and (2).

~~Unless indicated otherwise, our notation will be the same as used by Frucht.~~

2. Inequalities of Hardy-Littlewood-Pólya (H-L-P) — If $a_1, a_2, a_3, \dots a_n$ be n positive numbers, and ~~if~~ c_r ~~is~~ the r^{th} elementary symmetric function of the a, i.e. the sum of the products, r at a time, of different a, and p_r the average of these products, or, in the notation for permutations, if

$$c_r = \frac{1}{r!(n-r)!}\sum ! \, a_1 a_2 \cdots a_r,$$

and
$$p_r = \frac{r!(n-r)!}{n!} c_r, \quad \text{with } c_0 = p_0 = 1,$$

the inequalities in question are given by

$$p_{r-1}\, p_{r+1} < p_r^2 \quad (1 \leqq r < n) \qquad \dots\dots (2.1)$$

unless all the a are equal

In the present paper, we are interested only in the case $n = 3$, so that there are only two inequalities of the type (1) which can be written as,

$$(2.2) \quad \left.\begin{array}{l} \cancel{p_1^2 > p_2} \;\; p_1^2 - p_2 > 0 \\ \text{or } c_1^2 - 3c_2 > 0 \end{array}\right\} \dots\dots$$

and,
$$(2.3) \quad \left.\begin{array}{l} p_2^2 - p_1 p_3 > 0 \\ \text{or } c_2^2 - 3c_1 c_3 > 0 \end{array}\right\} \dots\dots$$

using $c_1 = 3p_1$; $c_2 = 3p_2$; $c_3 = p_3$.

(2.2) (2.3)

The inequalities (2.2) and (2.3) which, of course, follow from the general inequality (2.1) of H-L-P, can also be seen to be immediate consequences of the following identities which ~~can be easily verified;~~ easily verifiable identities,

(2.4) $$\sum (a_2-a_3)^2 \equiv 2(c_1^2-3c_2) \equiv 18(p_1^2-p_2)$$

(2.5) and $$\sum a_1^2(a_2-a_3)^2 \equiv 2(c_2^2-3c_1c_3) \equiv 18(p_2^2-p_1p_3)$$

The expressions $(p_1^2-p_2)$ and $(p_2^2-p_1p_3)$ being of different degrees are not comparable, but we can obtain an inequality ~~obtaining~~ involving them by means of the identity

(2.6) $$\sum a_2a_3(a_2-a_3)^2 \equiv c_2(c_1^2-3c_2)-(c_2^2-3c_1c_3) = 9\{3p_2(p_1^2-p_2)-(p_2^2-p_1p_3)\}$$

leading to

(2.7) $$\left.\begin{aligned} c_2(c_1^2-3c_2) &> c_2^2-3c_1c_3 \\ \text{or, } 3p_2(p_1^2-p_2) &> p_2^2-p_1p_3 \end{aligned}\right\}$$

~~We show elsewhere that (2.7) can be generalised to arbitrary n and r.~~ Finally we ~~place~~ ~~note also~~ note also the identity (2.8) $\Sigma a_1(a_2-a_3)^2 \equiv c_1c_2-9c_3 = 9(p_1p_2-p_3)$.

3) <u>Gerretsen's inequalities</u> – These are

(3.1) $$\Delta\sqrt{3}+\tfrac{1}{2}Q \geqq 4Rr+r^2 \geqq \Delta\sqrt{3}$$

where R, and r are the radii of the circum- and in-circles, Q is the measure of the "inequilaterality" or "scalenity" of the triangle given by

(3.2) $$Q = (b-c)^2+(c-a)^2+(a-b)^2,$$

where a, b, c are the sides of the triangle. A little simplification shows that (3.1) can be written in the equivalent form

(3.3) $$K-H \geqq 2\sqrt{3}\,\Delta \geqq 3K-5H, \qquad (3K > 5H)$$

where, in the notation of Bealty,

(3.4) $$H = \tfrac{1}{2}(a^2+b^2+c^2), \quad K = bc+ca+ab.,$$ and the equality on both sides of (3.3) holding only when the triangle is equilateral.

To prove (3.3), we note that we can take $a > b > c$ without loss of generality, and put

(3.5) $$a_1 = b+c-a, \quad a_2 = c+a-b, \quad a_3 = a+b-c$$

thereby getting a_1, a_2, a_3 as positive numbers. Noting that

(3.6) $$16\Delta^2 = c_1c_3, \quad \cancel{K = c_2 \text{ and } H = \tfrac{1}{2}(c_1^2-2c_2)}, \text{ and } K-H = \tfrac{1}{2}c_2$$

it is immediately seen that (2.3) gives the left-hand side of (3.3). The right hand

~~side~~

3. Beatty's inequalities – These are given by

$$(3.1) \qquad \frac{(K-H)^2}{12} \geqslant \Delta^2 \geqslant \frac{(K-H)(3K-5H)}{12}, \text{ (with } 3K > 5H\text{), where}$$

~~where~~

$$(3.2) \qquad H = \tfrac{1}{2}(a^2+b^2+c^2), \quad K = bc + ca + ab,$$

$\underline{a}$, $\underline{b}$, and $\underline{c}$ being the sides of the triangle, and both the equality sign holding only for an equilateral triangle.

To prove (3.1) we notice that we can take $a > b > c$ without any loss of generality, and put

$$(3.3) \qquad a_1 = b+c-a, \quad a_2 = c+a-b, \quad a_3 = a+b-c,$$

thereby getting a_1, a_2, a_3 as positive numbers. Noting that

$$(3.4) \qquad 16\Delta^2 = c_1c_3 \,;\, 4H = c_1^2 - c_2 \,;\, 4K = c_1^2 + c_2,$$

we see immediately that the left hand side of the inequality (3.1) ~~in an~~ a follows from (2.3), and the right hand side follows from (2.7).

4. Frucht's inequalities – These are given by

$$(4.1) \qquad \frac{s(s-q)^2(s+2q)}{27} \geqslant \Delta^2 \geqslant \frac{s(s+q)^2(s-2q)}{27}, \quad (s > 2q), \text{ where}$$

$$(4.2) \qquad q = (a^2+b^2+c^2 - bc - ca - ab)^{1/2}; \quad 2s = a+b+c, \text{ and}$$

the first and second) equality signs in (4.1) holding for isosceles triangles ~~whose~~ with $a = b$, and $b = c$ respectively.

To prove (4.1) is equivalent to proving

$$(4.3) \qquad \left|\Delta^2 - \frac{s^2}{27}(s^2-3q^2)\right| \leq \frac{2}{27}sq^2, \quad \text{(Equation (2.19) of Frucht's article)}$$

~~as has~~ and noting that

$$(4.4) \qquad 4q^2 = c_1^2 - 3c_2, \text{ and } s = \tfrac{1}{2}c_1, \text{ and } 16\Delta^2 = c_1c_3,$$ ~~(4.3) is~~ it can be shown after a little simplification that (4.3) is equivalent to

$$(4.5) \qquad 4(c_1^2-3c_2)(c_2^2-c_1c_3) > (c_1c_2-9c_3)^2.$$

Using (2.4), (2.5) and (2.8), this is equivalent to proving that

$$(4.6) \qquad \{\Sigma(a_2-a_3)^2\}\{\Sigma a_1^2(a_2-a_3)^2\} > \{\Sigma a_1(a_2-a_3)^2\}^2,$$ which is nothing but Cauchy's well-known inequality $\Sigma x^2 \cdot \Sigma y^2 > (\Sigma xy)^2$ (See H-L-P, Theorem 7, p.16)

5. Inequalities ~~for obtuse and acute angled~~ reducing to equalities for right angled triangles — We set

$$(5.1)\quad E = \frac{1}{2}\left\{(K+H) - \sqrt{2H(K+H)}\right\} = s^2 - \sqrt{s^2(2s^2+q^2)/3}\,,$$ ~~with~~ the ~~sign of the~~ square root having the positive value, and prove the following ~~theorems~~ results:

(a) ~~Theorem 1~~ — If $a^2 > b^2 + c^2$, ie $A > 90°$, then

~~(5.2)~~ $E > \Delta$, and E is nearer to Δ than the upper limits of ~~both~~ Beatty and Frucht.

(b) ~~Theorem 2~~ — If $a^2 < b^2 + c^2$, ie $A < 90°$, then

~~(5.3)~~ $E < \Delta$, and E is nearer to Δ than the lower limits of ~~both~~ Beatty and Frucht, under certain conditions.

(c) ~~Theorem 3~~ — If $a^2 = b^2 + c^2$ ie $A = 90°$, then $E = \Delta$

~~(5.4)~~ ~~$E = \Delta$.~~

Before proceeding to prove these results, we might notice at once that $E > 0$, since $K > H$ as is evident from (3.4) or $s > q$ as shown by (4.4). Also proof of 5(c) is immediate, for, when $a^2 = b^2 + c^2$, $H = a^2$, and with $K + H = 2s^2$

$$E = \frac{1}{2}\left\{2s^2 - \sqrt{4s^2a^2}\right\} = s(s-a) = \frac{1}{4}(a+b+c)(b+c-a)$$

$$= \frac{1}{4}\left\{(b+c)^2 - a^2\right\} = \frac{1}{2}bc = \Delta,$$ the triangle being right-angled with $A = 90°$.

To prove the first parts of 5(a) and 5(b), we write, using (3.4)

$$E = \frac{1}{4}c_1\left(c_1 - \sqrt{c_1^2 - c_2}\right), \text{ and } \Delta = \frac{1}{4}\sqrt{c_1c_3}, \text{ so that}$$

$E \gtrless \Delta$ according as $c_1\left(c_1 - \sqrt{c_1^2 - c_2}\right) \gtrless \sqrt{c_1c_3}$, and on simplification reduces to the result:

$$(5.2)\quad E \gtrless \Delta \text{ according as } (c_1c_2 + c_3)^2 - 4c_1^3c_3 \gtrless 0.$$ A little calculation enables us to show that

$$(5.3)\quad (c_1c_2 + c_3)^2 - 4c_1^3c_3 \equiv 8(a^2 - b^2 - c^2)(b^2 - c^2 - a^2)(c^2 - a^2 - b^2).$$

In view of $a > b > c$, the last two factors on the right-hand side of (5.3) are negative, and hence it follows from (5.2) and (5.3) that

$$E \gtrless \Delta \text{ according as } a^2 \gtrless b^2 + c^2.$$

To prove the second parts of 5(a) and 5(b), it is enough if we compare E with Frucht's upper & limits ~~since~~ only, since these are always closer to Δ than Beatty's limits. We have therefore to compare E with Frucht's limits

$$\frac{(s-q)\sqrt{s(s+2q)}}{3\sqrt{3}}, \text{ and } \frac{(s+q)\sqrt{s(s-2q)}}{3\sqrt{3}}$$ which we shall denote by F_1 and F_2 respectively. Using the second expression for E given in (5.1), and putting $\frac{s}{q} = x$, with $x > 1$, since $s > q$ as remarked earlier, it can be shown after some simplification that the condition $E < F_1$ or $F_1 > E$ is equivalent to the inequality

$$(5.4) \quad 2x^6 - 21x^4 + 44x^3 - 36x^2 + 12x - 1 > 0,$$

which reduces to

$$(5.4a) \quad (x-1)^4(2x^2 + 8x - 1) > 0;$$

and this is obviously true, the second factor being > 0 for $x > 1$. It may be noted that the condition $E < F_1$ holds irrespective of $a^2 >$ or $< b^2 + c^2$, but the condition $a^2 > b^2 + c^2$ is necessary to make $E > \Delta$, and thus show that E is nearer to Δ than F_1.

It can be shown in exactly a similar manner that the inequality $E > F_2$ or $F_2 < E$ is equivalent to

$$(5.5) \quad 2x^6 - 21x^4 - 44x^3 - 36x^2 - 12x - 1 < 0$$

which reduces to

$$(5.5, a) \quad (x+1)^4(2x^2 - 8x - 1) < 0.$$

In this case, both the inequalities of Beatty and Frucht are of no value unless $3K > 5H$ or $s > 2q$. We therefore need consider (5.5, a) only for $x > 2$. The quadratic factor in (5.5, a) is not negative for all $x > 2$, so that unlike as in the case of (5.4, a), ~~the inequality $E > F_2$~~ a further condition is required in order that the inequality $E > F_2$ may be satisfied. This condn can be easily found, and is given by ~~written~~ as

$$x < 2 + \frac{3\sqrt{2}}{2}, \text{ that the condition for } E > F_2 \text{ is}$$

$$(5.6) \quad 2q < s < q\left(2 + \frac{3\sqrt{2}}{2}\right)$$

If (5.6) is satisfied in addition to $a^2 < b^2 + c^2$, then E is nearer to Δ than the lower limits of Beatty and Frucht. It is however possible to find a condition analogous to (5.6) so that E is nearer to Δ than Beatty's lower limit, but not Frucht's. Using (3.1) this condition can be written as

$$\frac{\sqrt{(K-H)(3K-5H)}}{2\sqrt{3}} < E,$$ and this reduces to $9K^2 - 14KH - 7H^2 < 0$, and, when expressed in terms of s and q, to $s < q(3 + \sqrt{7})$. Denoting the upper and lower limits of Beatty by B_1 and B_2, we thus see that the condition $E > B_2$ is equivalent to

$$(5.7) \quad 2q < s < q(3 + \sqrt{7}).$$

Hence for a value of s greater than $q\left(2 + \frac{3\sqrt{2}}{2}\right)$ but less than $q(3 + \sqrt{7})$, E lies between

B_2 and F_2. In view of the fact that (5.6) and (5.7) are inequalities for the ratio of two symmetric functions of the sides, it does not appear possible to give them any definite geometric interpretation. We shall consider, however, ~~(5.6)~~ only (5.6) which compares E with F_2, and ~~since~~ find out the types of <u>isosceles</u> acute angled triangles ~~(with $a^2 < b^2 + c^2$~~ ($a > b > c$, and $a^2 < b^2 + c^2$) for which E is nearer to Δ than F_2.

We will show first that there exist no such triangles with $b = c$. For, in this case $2s = a + 2b$, and $q = a - b$, and ~~(5.6)~~ the second part of the inequality (5.6) reduces to

$$(5.8) \qquad b = c < \frac{a}{\sqrt{2}}$$

so that $b^2 + c^2 < a^2$, and the triangle is not acute-angled. We may also note that the limiting case of the right-hand side of (5.6) reducing to an equality gives $b = c = \frac{a}{\sqrt{2}}$ i.e. the triangle is right-angled with $A = 90^\circ$, $B = C = 45^\circ$, and $E = F_2$.

Next considering triangles with $a = b$, we note at once that the condition $a^2 < b^2 + c^2$ is satisfied. The left-hand side of (5.6) requires $a = b < \frac{5c}{2}$, while the right-hand side can be reduced to

$$(5.9) \qquad a = b > \frac{(8 + 9\sqrt{2})c}{14} \simeq 1.48c.$$

~~Thus~~, in particular, for all isosceles triangles with $a = b$, and $3c < 2b < 5c$, we have E is greater than F_2 and nearer to Δ than F_2. Three numerical examples, ~~may be cited~~, correct to two decimals places, are appended below in tabular form.

Delete

(i) $a = 3$, $b = 3$, $c = 2$ giving, to two decimal places, $F_2 = 2.72$, $E = 2.73$, $\Delta = 2.83$;

(ii) $a = 4$, $b = 4$, $c = 2$ giving $F_2 = 3.35$, $E = 3.79$, $\Delta = 3.87$;

(iii) $a = 9$, $b = 9$, $c = 4$ giving $F_2 = 10.21$, $E = 17.23$, $\Delta = 17.55$

$a = b$	c	b/c	F_2	E	Δ
3	2	1.5	2.72	2.73	2.83
4	2	2	3.35	3.79	3.87
9	4	2.25	10.21	17.23	17.55

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